# NORMS OF POWERS OF ABSOLUTELY CONVERGENT FOURIER SERIES

G. W. Hedstrom

## 1. INTRODUCTION

For an absolutely convergent Fourier series

$$f(t) = \sum c_k e^{ikt}$$

we use the norm

$$\|f\| = \sum |c_k|,$$

and we seek estimates of $\|f^n\|$ as $n \to \infty$. Such estimates are important for the study of the behavior of the solution of the difference equation

$$(1.1) \qquad v_k^{n+1} = \sum_j c_j v_{k-j}^n \qquad (n = 1, 2, \cdots; k = 0, \pm 1, \cdots)$$

with preassigned values $v_k^0$ $(k = 0, \pm 1, \pm 2, \cdots)$. This difference equation is sometimes used to approximate a hyperbolic or parabolic partial differential equation (see [9], [11]). The solution of equation (1.1) can be written in the form

$$v_k^n = \sum_j c_{jn} v_{k-j}^0 \qquad (n = 1, 2, \cdots),$$

where $c_{kn}$ is determined by the formula

$$f^n(t) = \sum_k c_{kn} e^{ikt} \qquad (n = 1, 2, \cdots).$$

Thus we have the inequality

$$\sup_k |v_k^n| \leq \|f^n\| \sup_k |v_k^0|,$$

and equality is attained if $v_k^0 = \exp\{i \arg c_{-k,n}\}$.

The behavior of $\|f^n\|$ as $n \to \infty$ has been the subject of several investigations. Beurling [7, pp. 428-429] proved that $\lim \|f^n\|^{1/n} = \max |f|$ (since f is necessarily continuous, max $|f|$ exists). Beurling and Helson [1] and Leĭbenzon [6] proved that if $|f(t)| \equiv 1$ and $\|f^n\|$ is bounded, then $f(t) = \exp\{i(a_0 + kt)\}$ for some integer k and some real $a_0$. Also, it follows easily from their work that if $|f(t)| = 1$ on a set S of positive measure, $|f(t)| < 1$ on the complement of S, and $\|f^n\|$ is bounded, then $f(t) = \exp\{i(a_0 + kt)\}$, where $a_0$ is real and k is an integer. (The author is indebted

Received January 26, 1966.

to the referee and to J. Korevaar for this remark.) In fact, with the notation $g_n = |f^n|^2$, the norms $\|g_n\|$ are bounded and

$$\lim g_n(t) = \begin{cases} 1 & (t \in S), \\ 0 & (t \notin S). \end{cases}$$

But it follows from Lebesgue's dominated-convergence theorem that a punctual limit of functions of bounded norms has a finite norm. Therefore S either has measure zero or is the interval $[0, 2\pi]$.

Before describing more precise estimates of $\|f^n\|$, we introduce some conditions.

(A) We say that f satisfies condition (A) on an interval $T = [t_1, t_2]$ if $f = e^{i\phi}$, where $\phi$ is real and piecewise linear on T.

(B) We say that condition (B) holds on T if $|f(t)| = 1$ on T and there exist a $t_0 \in T$ and a $\delta > 0$ such that if $t_0 \leq t \leq t_0 + \delta < t_2$, then

$$f(t) = \exp\{i(a_0 + a_1(t - t_0) + a_2(t - t_0)^2 + v(t))\}, \tag{1.2}$$

where $a_2 \neq 0$, $a_0$, $a_1$, $a_2$ are real, and v is a real-valued function in $C^3$ satisfying inequalities of the form

$$|v(t)| \leq C(t - t_0)^3, \quad |v'(t)| \leq C_1(t - t_0)^2, \quad |v''(t)| \leq C_2(t - t_0) \quad (t_0 \leq t \leq t_0 + \delta). \tag{1.3}$$

(C) Condition $(C_R)$ (R for "right") is said to hold at $t_0$ if $|f(t_0)| = 1$ and if, for some $\delta > 0$,

$$f(t) = \exp\{ia_0 + ia_1(t - t_0) - \gamma(t - t_0)^N + v(t)\} \quad (t_0 \leq t \leq t_0 + \delta), \tag{1.4}$$

where N is a positive integer, $a_0$ and $a_1$ are real, $\Re\gamma > 0$ (if $N = 1$, we require that $\gamma > 0$), and v is a function of bounded variation satisfying inequalities of the form

$$\begin{cases} |v(t)| \leq C(t - t_0)^{N+1} & (t_0 \leq t \leq t_0 + \delta), \\ \operatorname*{var}_{[t_0, t_0+\alpha]} v \leq C_1 \alpha^{N+1} & (0 < \alpha < \delta). \end{cases} \tag{1.5}$$

(D) Condition $(D_R)$ is said to hold at $t_0$ if $|f(t_0)| = 1$ and if there exist a $\delta > 0$ and integers M and N $(2 \leq M < N)$ such that

$$f(t) = \exp\left\{ia_0 + ia_1(t - t_0) + i\sum_M^N a_j(t - t_0)^j - \gamma(t - t_0)^N + v(t)\right\} \quad (t_0 \leq t \leq t_0 + \delta),$$

where $a_j$ is real $(j = 0, 1, M, M + 1, \cdots, N)$, $a_M \neq 0$, $\gamma > 0$, and v satisfies (1.5).

Conditions $(C_L)$ and $(D_L)$ are analogous to $(C_R)$ and $(D_R)$, the only difference being that they are imposed to the left of $t_0$.

*Remark.* When we say that an estimate of the form (1.2), (1.4), or (1.6) holds in neighborhoods of two different points or to the right and left of one point, we do not mean to imply that the N, M, $a_j$, v, and so forth are the same.

THEOREM. *Let* f *be absolutely continuous on the circle, and let* f' *be of bounded variation. Let* $|f(t)| \leq 1$, *and let* $|f(t)| = 1$ *on a finite number of disjoint closed intervals* $I_j$ $(j = 1, 2, \cdots, J;\ J \geq 0)$ *and at a finite number of isolated points* $t_m$ $(m = 1, 2, \cdots, \mu;\ \mu \geq 0)$. *On each* $I_j$ $(j = 1, \cdots, J)$, *let either condition* (A) *or* (B) *hold. Let a condition* (C) *or* (D) *hold at each end of an interval* $I_j$ *and at both sides of each* $t_m$. *Then there exist positive constants* C, C' *such that*

$$C'\,g(n) < \|f^n\| < C\,g(n) \qquad (n = 2, 3, \cdots), \tag{1.7}$$

*where* g(n) *is determined in the following way.*

(i) *If* (B) *holds on some interval, then* $g(n) = \sqrt{n}$.

(ii) *If* (B) *never holds but* (D) *holds at least once, then* $g(n) = n^b$, *where* $b = \max(N - M)/2N$, *the maximum being taken over the finite set of points where* (D) *holds.*

(iii) *If* (B) *and* (D) *never hold and if* $J > 0$ *(so that* (A) *holds) but* $|f(t)| < 1$ *for some* t, *or if* $J = 0$ *and* $\mu > 0$ *and there is a point* $t_m$ *such that at least one of the numbers* $a_1$ *and* N *on the right of* $t_m$ *differs from the corresponding* $a_1$ *or* N *on the left, then* $g(n) = \log n$. *Also,* $g(n) = \log n$ *if* $f = e^{i\phi}$, *with* $\phi$ *piecewise linear but not linear.*

(iv) *If* $J = 0$ *and* $\mu > 0$, *and if* (C) *holds at each* $t_m$ *in such a way that on both sides of* $t_m$ *either* $N = 1$ *or* $N > 1$ *and the* $a_1$, N *are the same, then* $g(n) = 1$.

*Remarks.* The upper estimate in case (i) is due to Kahane (see [5, p. 103]), who also proved the lower estimate for analytic f [4]. We shall show in Section 6 that the lower estimate is valid under our weaker conditions.

Case (ii) has been considered by several authors. Serdjukova [8] proved that if f is analytic, then

$$\|f^n\| > C\,n^{1/2(M+1)-\varepsilon}.$$

Thomée [11] proved that under the conditions of the theorem,

$$\|f^n\| > C\,n^{1/M-1/N}.$$

Hedstrom [3] proved inequality (1.7) when $N = M + 1$ and f is analytic. Most of the present paper is devoted to case (ii).

Kahane [4] has discussed (iii) when $f = e^{i\phi}$ and $\phi$ is piecewise linear on $[0, 2\pi]$. The estimate for case (iv) is due to Strang [10].

Finally, we remark that the behavior of $\|f^n\|$ is determined by the local behavior of f. We use a smooth partition of unity

$$1 = \sum_{1}^{J+\mu} p_j(t),$$

where $p_j(t) = 1$ on a neighborhood of one and only one component of S. Then localization follows from the inequality

$$\frac{\|p_j f^n\|}{\|p_j\|} \le \|f^n\| \le \sum_{j=1}^{J+\mu} \|p_j f^n\| \qquad (n = 1, 2, \cdots).$$

## 2. OUTLINE OF THE PROOF OF THE THEOREM

We shall make estimates of

$$c_{kn} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^n(t)\, e^{-ikt}\, dt$$

for large $n$ and for $-\infty < k < \infty$. We first consider large values of $|k|$.

LEMMA 2.1. *Let* $f$ *be absolutely continuous on the circle, and let* $f'$ *be of bounded variation. Let* $|f(t)| \le 1$ *for all* $t$. *Then there exist constants* $C$ *and* $\mu$ *such that*

$$\sum_{|k| \ge \mu n} |c_{kn}| \le C \qquad (n = 1, 2, \cdots).$$

*Proof.* Since $f$ is continuous and $f'$ is essentially bounded, it is possible to divide the circle into sets $R$ and $S$, each a finite collection of intervals, such that $|f(t)| > 1/3$ for $t$ in $R$ and $|f(t)| < 2/3$ for $t$ in $S$. Let $(t_1, t_2)$ be an interval in $R$, and define the logarithm so that $g(t) = \log f(t)$ is continuous on this interval. Then, by integrating by parts twice on the right side of the equation

$$\int_{t_1}^{t_2} f^n(t)\, e^{-ikt}\, dt = \int_{t_1}^{t_2} \exp\{ng(t) - ikt\}\, dt,$$

we obtain the inequality

$$\left| \int_{t_1}^{t_2} f^n(t)\, e^{-ikt}\, dt - \left[\frac{f^n(t)\, e^{-ikt}}{ng'(t) - ik}\right]_{t_1}^{t_2} \right| \le \operatorname*{var}_R \frac{1}{ng' - ik}.$$

Note that $g$ is absolutely continuous and $g'$ is of bounded variation, because $f$ has these properties. If we require that

$$|k| > 2n \operatorname*{ess\,sup}_R |g'| = 2n \operatorname*{ess\,sup}_R \left|\frac{f'}{f}\right| = \mu n, \tag{2.1}$$

then

$$\operatorname*{var}_R \frac{1}{ng' - ik} \le \frac{Cn}{k^2} \operatorname*{var}_R g'.$$

Consequently, if $(t_1, t_2) \subset R$ and $k$ satisfies (2.1), then

$$\left| \int_{t_1}^{t_2} f^n(t)\, e^{-ikt}\, dt - \left[\frac{f^n(t)\, e^{-ikt}}{ng'(t) - ik}\right]_{t_1}^{t_2} \right| \le \frac{Cn}{k^2} \qquad (n = 1, 2, \cdots). \tag{2.2}$$

Let $(t_1, t_2)$ be an interval in S. Two consecutive integrations by parts of $\int_{t_1}^{t_2} f^n(t)\,e^{-ikt}\,dt$ give the inequality

$$\left| \int_{t_1}^{t_2} f^n(t)\,e^{-ikt}\,dt - \left.\frac{f^n(t)\,e^{-ikt}}{-ik}\right]_{t_1}^{t_2} \right| \leq \frac{Cn}{k^2} \operatorname*{var}_S (f' f^{n-1}) \leq \frac{Cn}{k^2}\left(\frac{2}{3}\right)^n \tag{2.3}$$

$$(n = 1, 2, \cdots).$$

Denote the endpoints of the intervals in R and S by $t_j$ $(j = 1, 2, \cdots, J)$. From (2.2) and (2.3) it follows that if k satisfies (2.1), then

$$2\pi\,|c_{kn}| \leq \sum_{j=1}^{J} |f^n(t_j)| \left| \frac{1}{ng'(t_j) - ik} + \frac{1}{ik} \right| + \frac{Cn}{k^2} \leq \frac{C'n}{k^2} \qquad (n = 1, 2, \cdots).$$

Therefore $\sum_{|k| \geq \mu n} |c_{kn}| \leq C$ $(n = 1, 2, \cdots)$, and the lemma is proved.

If the condition $(C_R)$ or $(D_R)$ holds at $t = 0$ and if $|f(t)| < 1$ for $0 < t \leq \pi$, then, for $|k| \leq \mu n$, we use the decomposition

$$\int_0^{\pi} f^n(t)\,e^{-ikt}dt = \int_0^{\alpha_n} f^n(t)\,e^{-ikt}dt + \int_{\alpha_n}^{\pi} f^n(t)\,e^{-ikt}dt, \tag{2.4}$$

where $\alpha_n = n^{-1/(N+1)}$. It follows from (1.4) or (1.6) that if n is large, then

$$\left| \int_{\alpha_n}^{\pi} f^n(t)\,e^{-ikt}dt \right| \leq \pi\,|f^n(\alpha_n)| \leq \pi \exp\{-\Re\gamma'\,n^{1/(N+1)}\} \qquad (0 < \gamma' < \Re\gamma). \tag{2.5}$$

If $(D_R)$ holds at $t = 0$, we split the first integral on the right in (2.4) into two integrals:

$$\left\{\begin{aligned} \int_0^{\alpha_n} f^n(t)\,e^{-ikt}\,dt &= \int_0^{\alpha_n} \exp\{-i(k - na_1)t + in \sum_M^N a_j t^j - n\gamma t^N\}\,dt \\ &\quad + \int_0^{\alpha_n} [e^{nv(t)} - 1]\,e^{-n\gamma t^N} \exp\left(i\left[-(k - na_1)t + n \sum_M^N a_j t^j\right]\right)dt \\ &= a_n(w) + b_n(w) \qquad \left(w = \frac{k}{n} - a_1\right). \end{aligned}\right. \tag{2.6}$$

We use complex-variable methods to estimate $a_n(w)$, and the method of stationary phase to estimate $b_n(w)$.

If $(C_R)$ holds at $t = 0$ and $|f(t)| < 1$ $(0 < t \leq \pi)$, we write

$$
(2.7)\quad \begin{cases} \displaystyle\int_0^{\alpha_n} f^n(t)\, e^{-ikt}\, dt = \int_0^{\alpha_n} \exp\{-i(k - na_1)t - n\gamma t^N\}\, dt \\ \displaystyle\qquad + \int_0^{\alpha_n} [e^{nv(t)} - 1]\, e^{-n(\Re\gamma)t^N} \exp\{i(-(k - na_1)t + n\,\Im\,\gamma t^N)\}\, dt \\ \displaystyle\qquad = A_n(w) + B_n(w) \qquad \left(w = \frac{k}{n} - a_1\right). \end{cases}
$$

The methods that are used to estimate $a_n(w)$ and $b_n(w)$ also give us estimates of $A_n(w)$ and $B_n(w)$ as $n \to \infty$ ($|k| \leq \mu n$).

We shall examine $a_n(w)$ in the next section. Then, in Section 4, we discuss $b_n(w)$ and $B_n(w)$. In Section 5 we prove the theorem for cases (ii), (iii), and (iv). In Section 6 we use similar methods to prove case (i).

## 3. ESTIMATES OF $a_n(w)$ AND $A_n(w)$

Consider $a_n(w) = \int_0^{\alpha_n} \exp n\phi(t)\, dt$, where $\alpha_n = n^{-1/(N+1)}$ and

$$
(3.1)\qquad \phi(t) = -iwt + i \sum_M^N a_j t^j - \gamma t^N
$$

for some integers $M$ and $N$ ($2 \leq M \leq N$), for $\gamma > 0$, for $a_M > 0$, for real values $a_j$ ($j = M + 1, \cdots, N$), and for a real parameter $w$ ($-\infty < w < \infty$). The methods used to estimate $a_n(w)$ as $n \to \infty$ will also be used to estimate

$$
A_n(w) = \int_0^{\alpha_n} \exp n\psi(t)\, dt
$$

as $n \to \infty$, where

$$
(3.2)\qquad \psi(t) = -iwt - \gamma t^N
$$

($-\infty < w < \infty$, $N$ a positive integer, $\Re\gamma > 0$, $\Im\gamma \leq 0$; if $N = 1$, we require that $\Im\gamma = 0$).

For $a_n(w)$, the most important region is

$$
(3.3)\qquad \lambda(n)\, n^{-(M-1)/M} \leq w \leq n^{-(M-1)/(N+1)},
$$

where

$$
(3.4)\qquad \lambda(n) = (\log n)^{3(M-1)/M}.
$$

LEMMA 3.1. *In* (3.1), *let* $w$ *satisfy* (3.3). *Then for some nonzero complex constant* $C_0$ *depending on* $M$ *and* $a_M$, *the relation*

$$
\begin{aligned}
a_n(w) &= -\frac{i}{nw} + O(n^{-M} w^{-(M+1)}) \\
&\qquad + C_0\, n^{-1/2} w^{-(M-2)/2(M-1)} \exp\{n\phi(t_0)\}\{1 + O((\log n)^{-3/4})\}
\end{aligned} \tag{3.5}
$$

*holds uniformly with respect to* w *as* $n \to \infty$. *As* $w \to 0+$,

$$
\left\{
\begin{aligned}
\Re\,\phi(t_0) &= -\gamma\left(\frac{w}{Ma_M}\right)^{N/(M-1)} + O(w^{(N+1)/(M-1)}), \\
\Im\,\phi(t_0) &= -(Ma_M)^{-1/(M-1)}\left(1 - \frac{1}{M}\right) w^{M/(M-1)} + O(w^{(M+1)/(M-1)}).
\end{aligned}
\right. \tag{3.6}
$$

*Proof.* Since $\phi$ is an entire function, we may deform the contour so that it passes through a saddle point of $\exp n\phi(t)$. Thus we seek a point $t_0$ such that $\phi'(t_0) = 0$. We find that as $w \to 0+$, the saddle we want is given by

$$
t_0 = (1 + \varepsilon)\left(\frac{w}{Ma_M}\right)^{1/(M-1)}, \tag{3.7}
$$

where we have taken the positive $(M - 1)$th root, and where

$$
\begin{aligned}
\Re\,\varepsilon &= O(w^{1/(M-1)}), \\
\Im\,\varepsilon &= -\frac{\gamma N}{M-1}(Ma_M)^{-(N-1)/(M-1)} w^{(N-M)/(M-1)}\{1 + O(w^{1/(M-1)})\}
\end{aligned} \tag{3.8}
$$

as $w \to 0+$. Note that if w satisfies the inequality (3.3), then $0 < t_0 \le C\alpha_n$. Note also that the substitution of this estimate of $t_0$ yields (3.6).

We use a contour consisting of a segment $\Gamma_1$ from 0 to $-i|t_0|$, a segment $\Gamma_2$ of length $3|t_0|$ from $-i|t_0|$ through $t_0$ into the first quadrant, and a segment $\Gamma_3$ from the end of $\Gamma_2$ to $\alpha_n$. On $\Gamma_1$, the function $\exp n\phi(t)$ differs from $\exp\{-nw|t|\}$ by a function that is bounded by

$$
Cn\,|t|^M \exp\{-nw|t|\}.
$$

Thus, using the first inequality in (3.3), we obtain the estimate

$$
\int_{\Gamma_1} \exp n\phi(t)\,dt = \frac{-i}{nw} + O(n^{-M} w^{-(M+1)}) \qquad (n \to \infty). \tag{3.9}
$$

Expanding $\phi$ about $t_0$ we get the relation

$$
\phi(t) = \phi(t_0) + \sum_{2}^{N} \beta_j(w)(t - t_0)^j,
$$

where

$$
\beta_2(w) = \frac{i(M-1)}{2}(Ma_M)^{-1/(M-1)} w^{(M-2)/(M-1)} + O(w)
$$

as $w \to 0$. From estimates of the other $\beta_j(w)$ we find that if

$$\delta = (\log n)^{-1/4} n^{-1/3} w^{-(M-3)/3(M-1)},$$

then on the part of $\Gamma_2$ with $|t - t_0| < \delta$, $\exp n\phi(t)$ differs from

$$\exp\{n(\phi(t_0) + \beta_2(w)(t - t_0)^2)\}$$

by a function bounded by

$$C(\log n)^{-3/4} |\exp n(\phi(t_0) + \beta_2(w)(t - t_0)^2)| .$$

From the first inequality in (3.3), we conclude that

$$|\exp n\phi(t)| \leq \exp\{-C(\log n)^{1/2}\} \qquad (C > 0,\ n = 2, 3, \cdots)$$

on the remainder of $\Gamma_2$. Therefore, as $n \to \infty$,

$$\text{(3.10)} \qquad \int_{\Gamma_2} \exp n\phi(t)\,dt = e^{i\pi/4} \left(\frac{\pi}{n\,|\beta_2(w)|}\right)^{1/2} \exp n\phi(t_0)\,\{1 + O((\log n)^{-3/4})\}$$

if w satisfies (3.3).

On $\Gamma_3$,

$$|\exp n\phi(t)| \leq |\exp n\phi(\alpha_n)| \leq 2\exp\{-\gamma n^{1/(N+1)}\}$$

if n is large. Thus

$$\text{(3.11)} \qquad \left|\int_{\Gamma_3} \exp n\phi(t)\,dt\right| \leq \exp\{-\gamma n^{1/(N+1)}\} \qquad (n \text{ large}).$$

The estimate (3.5) now follows from (3.9), (3.10), and (3.11) if we take

$$C_0 = e^{i\pi/4} \left(\frac{2\pi}{M - 1}\right)^{1/2} (Ma_M)^{1/2(M-1)}.$$

This proves the lemma.

For $A_n(w)$ we have an analogous result, if

$$\text{(3.12)} \qquad \lambda_0(n)\,n^{-(N-1)/N} \leq w \leq n^{-(N-1)/(N+1)},$$

where

$$\text{(3.13)} \qquad \lambda_0(n) = (\log n)^{3(N-1)/N} .$$

LEMMA 3.2. *In* (3.2), *let* $N \geq 2$, $\gamma = re^{-i\theta}$ $(r > 0,\ 0 \leq \theta < \pi/2)$, *and let* w *satisfy* (3.12). *Then, as* $n \to \infty$,

$$\text{(3.14)} \qquad \begin{aligned} A_n(w) &= \frac{-i}{nw} + O(n^{-N} w^{-(N+1)}) \\ &\quad + Cn^{-1/2} w^{-(N-2)/2(N-1)} [\exp n\psi(t_0)]\,\{1 + O((\log n)^{-3/4})\} \end{aligned}$$

*uniformly in* w, *for some complex* $C \neq 0$. *Here*

$$(3.15)\qquad \psi(t_0) = -\beta w^{N/(N-1)}$$

*for some complex* $\beta$ *such that* $\Re\beta > 0$.

The proof of this lemma is similar to the proof of the previous one. The only difference is that the saddle point we want now is the solution

$$(3.16)\qquad t_0 = \left(\frac{w}{Nr}\right)^{1/(N-1)} \exp\left\{\frac{i}{N-1}\left(\theta - \frac{\pi}{2}\right)\right\}$$

of the equation $\psi'(t_0) = 0$. Equation (3.15) follows immediately from this. The rest of the proof consists of setting $M = N$ in the proof of Lemma 3.1.

We next consider small positive $w$.

LEMMA 3.3. *In equation* (3.1), *let*

$$(3.17)\qquad n^{-(M-1)/M} \leq w \leq \lambda(n)\, n^{-(M-1)/M},$$

*with* $\lambda(n)$ *as defined in* (3.4). *Then for some* C *independent of* n *and* w,

$$(3.18)\qquad |a_n(w)| \leq Cw^{1/(M-1)} \qquad (n = 1, 2, \cdots).$$

*Further, if*

$$(3.19)\qquad 0 \leq w \leq n^{-(M-1)/M},$$

*then for some* C *independent of* n *and* w,

$$(3.20)\qquad |a_n(w)| \leq Cn^{-1/M} \qquad (n = 1, 2, \cdots).$$

*Proof.* We again choose a contour through the saddle point $t_0$ given in equations (3.7), (3.8). Let $\Gamma_1$ consist of a segment from 0 to $-i\,|t_0|/2$ and a segment from there to $t_0$. Let $\Gamma_2$ consist of a segment from $t_0$ in the direction $\arg(t - t_0) = \pi/2M$ extending to $|t| = \alpha_n = n^{-1/(N+1)}$. Let $\Gamma_3$ be a segment from there to $\alpha_n$.

Just as in the proof of Lemma 3.1, we find that if n is large and w satisfies (3.17) or (3.19), then

$$(3.21)\qquad \left|\int_{\Gamma_3} \exp n\phi(t)\, dt\right| \leq \exp\{-\gamma n^{1/(N+1)}\}.$$

It is evident that $|\exp n\phi(t)| \leq 1$ on $\Gamma_1$ and that the length of $\Gamma_1$ is less than $4|t_0|$. Therefore it follows from (3.7) and (3.8) that if (3.17) or (3.19) holds, then

$$\left|\int_{\Gamma_1} \exp n\phi(t)\, dt\right| \leq Cw^{1/(M-1)} \qquad (n = 1, 2, \cdots).$$

On $\Gamma_2$, if (3.17) or (3.19) holds, we find that

$$(3.22)\qquad |\exp n\phi(t)| \leq |\exp n\phi(t_0)| \exp\{-bn|t - t_0|^M\} \leq \exp\{-bn|t - t_0|^M\}$$

for some b $(0 < b < a_M)$. Therefore, integrating (3.22), we get the inequality

$$\left|\int_{\Gamma_2} \exp n\phi(t)\,dt\right| \le Cn^{-1/M}.$$

Combining the above estimates, we find that if (3.17) or (3.19) holds, then

$$\left|\int_0^{\alpha_n} \exp n\phi(t)\,dt\right| \le C_1 w^{1/(M-1)} + C_2 n^{-1/M} \qquad (n = 1, 2, \cdots).$$

If (3.17) holds, then the first term on the right is the larger, giving (3.18). If (3.19) holds, then the second term is the larger, giving (3.20). The lemma is proved.

Our estimates for $A_n(w)$ for the corresponding $w$ are contained in the next two lemmas.

LEMMA 3.4. *In equation* (3.2), *let* $N \ge 2$, $\gamma = -re^{-i\theta}$ $(r > 0,\ 0 \le \theta < \pi/2)$, *and let*

$$n^{-(N-1)/N} \le w \le \lambda_0(n)\, n^{-(N-1)/N}, \tag{3.23}$$

*where* $\lambda_0(n)$ *is given by* (3.13). *Then*

$$\left|A_n(w) + \frac{i}{nw}\right| \le C n^{-N} w^{-(N+1)} \qquad (n = 1, 2, \cdots). \tag{3.24}$$

*Proof.* Let $t_0$ be defined as in (3.16). We use the following contour. Let $\Gamma_1$ be the segment from $0$ to $t_0$. Let $\Gamma_2$ be a segment from $t_0$ in the direction $\arg(t - t_0) = \theta/N$ extending to $|t| = \alpha_n$. Let $\Gamma_3$ be a segment from there to $\alpha_n$. If (3.23) holds, then, on $\Gamma_1$, $\exp n\psi(t)$ differs from $\exp\{-inwt\}$ by at most a constant multiple of

$$n\,|t|^N \exp\{-nwC\,|t|\} \qquad (n = 1, 2, \cdots),$$

for some $C > 0$. Thus if (3.23) holds, then

$$\int_{\Gamma_1} \exp n\psi(t)\,dt = -\frac{i}{nw} + O(n^{-N} w^{-(N+1)}) \qquad (n \to \infty).$$

Using (3.15) and the argument in the previous lemma, we find that if (3.23) holds, then

$$\left|\int_{\Gamma_2} \exp n\psi(t)\,dt\right| \le C n^{-1/N} \exp\{-bw^{N/(N-1)}\} \qquad (n = 1, 2, \cdots) \tag{3.25}$$

for some $b > 0$. Since the term on the right in (3.25) is bounded by $C' n^{-N} w^{-(N+1)}$ when (3.23) holds, and since

$$\left|\int_{\Gamma_3} \exp n\psi(t)\,dt\right| \le C\exp\{-\Re\gamma n^{1/(N+1)}\} \qquad (n = 1, 2, \cdots),$$

we find that inequality (3.24) holds. This proves the lemma.

LEMMA 3.5. *In* (3.1), *let* $w$ *satisfy the inequalities*

(3.26) $$-n^{-(M-1)/M} \leq w \leq 0.$$

*Then, for some* C *independent of* n,

(3.27) $$|a_n(w)| \leq C n^{-1/M} \quad (n = 1, 2, \cdots).$$

*Likewise, in* (3.2) *let* $N \geq 2$ *and* $|w| \leq n^{-(N-1)/N}$. *Then*

(3.28) $$|A_n(w)| \leq C n^{-1/N} \quad (n = 1, 2, \cdots),$$

*and there exist positive numbers* C' *and* ε *such that if* $|w| \leq \varepsilon n^{-(N-1)/N}$, *then*

(3.29) $$|A_n(w)| \geq C' n^{-1/N} \quad (n = 1, 2, \cdots).$$

*Proof.* Assume first that (3.26) holds. Let $\Gamma_1$ be a segment from 0 in the direction $\arg t = \pi/2M$ extending to $|t| = \alpha_n$. Let $\Gamma_3$ be a segment from there to $\alpha_n$. On $\Gamma_1$,

$$|\exp n\phi(t)| \leq \exp\{-bn|t|^M\} \quad (n = 1, 2, \cdots)$$

for some b $(0 < b < a_M)$. Thus, if (3.26) holds, then

$$\left|\int_{\Gamma_1} \exp n\phi(t)\right| \leq C n^{-1/M} \quad (n = 1, 2, \cdots).$$

The estimate (3.27) follows from this and from the fact that the integral over $\Gamma_3$ is small, as in (3.21).

To get estimates (3.28) and (3.29), we integrate

$$\exp n\psi(t) = (\exp\{-n\gamma t^N\})\{1 + O(n|w|t)\}$$

over the interval $[0, \alpha_n]$. This concludes the proof of the lemma.

We need one more set of estimates for $a_n(w)$ and $A_n(w)$ $(N \geq 2)$.

LEMMA 3.6. *In* (3.1), *let* w *satisfy either*

(3.30) $$w \leq -n^{-(M-1)/M}$$

*or*

(3.31) $$w \geq n^{-(M-1)/(N+1)}.$$

*Then there exists a* C *independent of* n *such that*

(3.32) $$\left|a_n(w) + \frac{i}{nw}\right| \leq C n^{-M} |w|^{-(M+1)} \quad (n = 1, 2, \cdots).$$

*Further, if in* (3.2) $N \geq 2$, $\gamma = -re^{-i\theta}$ $(r > 0,\ 0 \leq \theta < \pi/2)$, *and either*

(3.33) $$w \leq -n^{-(N-1)/N}$$

*or*

(3.34) $$w \geq n^{-(N-1)/(N+1)},$$

*then there exists a constant* C *independent of* n *such that*

$$\left|A_n(w) + \frac{i}{nw}\right| \le C n^{-N} |w|^{-(N+1)} \qquad (n = 1, 2, \cdots). \tag{3.35}$$

*Proof.* If (3.30) holds, choose a contour consisting of a segment $\Gamma_1$ from 0 to $|t| = \alpha_n$ in the direction $\arg t = \pi/2M$ and a segment $\Gamma_3$ from there to $\alpha_n$. If (3.31) holds, choose a contour consisting of a segment $\Gamma_1$ from 0 to $-i\alpha_n$ and a segment $\Gamma_3$ from there to $\alpha_n$ along the circle $|t| = \alpha_n$.

In both cases, we have the estimate (3.21) for the integral over $\Gamma_3$, and in both cases $\exp n\phi(t)$ differs from $\exp\{-inwt\}$ on $\Gamma_1$ by a function that is less than

$$C n |t|^M \exp\{-n |w| b |t|\} \qquad (n = 1, 2, \cdots),$$

for some $b > 0$. The estimate (3.32) follows from this.

If (3.33) holds, choose a contour consisting of a segment $\Gamma_1$, from 0 to $|t| = \alpha_n$ along the ray $\arg t = \theta/N$ and a segment $\Gamma_3$ from there to $\alpha_n$. If (3.34) holds, choose a segment $\Gamma_1$ from 0 to $-i\alpha_n$ and let $\Gamma_3$ be the arc of $|t| = \alpha_n$ from $-i\alpha_n$ to $\alpha_n$. Then the proof of (3.35) is the same as the proof of (3.32). This proves the lemma.

We are left with the case $N = 1$, that is, with the case $\psi(t) = -iwt - \gamma t$.

LEMMA 3.7. *In* (3.2), *let* $N = 1$ *and* $\gamma > 0$. *Then, for some* C *independent of* n,

$$\left|A_n(w) + \frac{i}{n(w + i\gamma)}\right| \le C e^{-\gamma\sqrt{n}} \qquad (n = 1, 2, \cdots). \tag{3.36}$$

The proof is trivial.

## 4. ESTIMATES OF $b_n(w)$ AND $B_n(w)$

We now need estimates of the quantities $b_n(w)$ and $B_n(w)$ defined in (2.6) and in (2.7). That is, let

$$b_n(w) = \int_0^{\alpha_n} h_n(t) \exp\{ing(t)\}\, dt, \tag{4.1}$$

where $\alpha_n = n^{-1/(N+1)}$,

$$g(t) = -wt + \sum_M^N a_j t^j; \tag{4.2}$$

here M and N are integers $(2 \le M < N)$, $a_j$ is real $(j = M + 1, \cdots, N)$, $a_M > 0$, and

$$h_n(t) = [e^{nv(t)} - 1] \exp\{-\gamma n t^N\}; \tag{4.3}$$

in the last formula, $\gamma$ is a positive constant and v is a continuous complex-valued function of bounded variation on $[0, \delta]$, for some $\delta > 0$ satisfying the conditions

$$|v(t)| \le C t^{N+1}, \qquad \operatorname*{var}_{[0,t]} v \le C' t^{N+1} \qquad (0 \le t \le \delta). \tag{4.4}$$

We shall use the method of stationary phase to prove the following lemma.

LEMMA 4.1. *In equation* (4.2), *let*

$$C_0\, n^{-(M-1)/(N+1)} \le |w| \,, \tag{4.5}$$

*where* $C_0$ *is a constant such that if* (4.5) *holds, then* $|g'(t)| \ge |w|/2$ *for* $0 \le t \le \alpha_n$. *Then, for* $0 < \varepsilon < 1/N$, *there exists a constant* $C_\varepsilon$ *such that*

$$|b_n(w)| \le C_\varepsilon\, n^{-1-\varepsilon}\, |w|^{-1} \qquad (n = 1, 2, \cdots). \tag{4.6}$$

*Further, if*

$$|w| \le C_0\, n^{-(M-1)/(N+1)}, \tag{4.7}$$

*then*

$$|b_n(w)| \le C_\varepsilon\, |w|^{(N+3-M)/2(M-1)}\, n^{-\varepsilon/2} \exp\left\{ \frac{-n\gamma}{2}\, |w|^{N/(M-1)} \right\}. \tag{4.8}$$

*Proof.* Assume first that (4.5) holds. Then integration by parts of (4.1) gives the estimate

$$\begin{aligned} |b_n(w)| &\le \left| \frac{h_n(\alpha_n)}{n\, g'(\alpha_n)} \right| + \operatorname*{var}_{[0,\,\alpha_n]} \frac{h_n}{n\, g'} \\ &\le \left| \frac{h_n(\alpha_n)}{n\, g'(\alpha_n)} \right| + \frac{1}{n} \sup |h_n| \operatorname{var} \frac{1}{g'} + \frac{1}{n} \sup \left| \frac{1}{g'} \right| \operatorname{var} h_n . \end{aligned} \tag{4.9}$$

If (4.5) holds, we see that for large $n$ (small $\alpha_n$)

$$\operatorname*{var}_{[0,\,\alpha_n]} \frac{1}{g'} \le \sup_{[0,\,\alpha_n]} \left| \frac{1}{g'} \right| \le \frac{2}{|w|}. \tag{4.10}$$

Also it follows from (4.3) and (4.4) that if $n$ is large, then

$$|h_n(t)| \le C n\, t^{N+1} \exp\{-n\gamma t^N\} \qquad (0 \le t \le \alpha_n). \tag{4.11}$$

Thus the first two terms on the right-hand side of (4.9) are bounded by the right-hand side of (4.6).

Now we need only estimate $\operatorname{var} h_n$. Let $\beta$ satisfy the inequalities $1/(N+1) < \beta < 1/N$. Then from (4.4) and (4.11) it follows that

$$\operatorname*{var}_{[0,\,\alpha_n]} h_n = \operatorname*{var}_{[0,\,n^{-\beta}]} h_n + \operatorname*{var}_{[n^{-\beta},\,\alpha_n]} h_n \le C n \cdot n^{-\beta(N+1)} + C' \exp\{-n\gamma n^{-N\beta}\}.$$

Consequently, if we put $\varepsilon = \beta(N+1) - 1$, we see that $0 < \varepsilon < 1/N$ and

$$\operatorname*{var}_{[0,\,\alpha_n]} h_n \le C_\varepsilon\, n^{-\varepsilon}.$$

This together with (4.10) proves that the third term on the right of (4.9) is also bounded by the right side of (4.6).

To get (4.8) under the assumption that (4.7) holds, we write

$$(4.12)\qquad b_n(w) = \left(\int_0^{t_0-\delta} + \int_{t_0-\delta}^{t_0+\delta} + \int_{t_0+\delta}^{\alpha_n}\right) h_n(t)\exp\{ing(t)\}\,dt.$$

Here $t_0 = 0$ if $w \leq 0$, and the first integral and the obvious part of the second are to be omitted. Let

$$0 < w \leq C_0 n^{-(M-1)/(N+1)}.$$

The equation $g'(\tau) = 0$ has a solution

$$\tau = \left(\frac{w}{Ma_M}\right)^{1/(M-1)} + O(w^{2/(M-1)}) \qquad (w \to 0+).$$

If $\tau < \alpha_n$, then $t_0 = \tau$; otherwise, $t_0 = \alpha_n$, and the third integral and part of the second are omitted.

The first and third integrals in (4.12) are estimated just as in the proof of the first part of the lemma. The only difference is that in place of (4.10) we have on $[0, t_0 - \delta]$ and $[t_0 + \delta, \alpha_n]$ ($n$ large) the inequalities

$$\operatorname{var}\frac{1}{g'} \leq \sup\left|\frac{1}{g'}\right| \leq \frac{C}{\delta\,|g''(\tau)|} \leq C'\delta^{-1}|w|^{-(M-2)/(M-1)}.$$

Thus if (4.7) holds, the first and third integrals in (4.12) are bounded by

$$(4.13)\qquad Cn^{-(1+\varepsilon)}\delta^{-1}|w|^{-(M-2)/(M-1)} \qquad (n = 1, 2, \cdots).$$

The second integral in (4.12) is bounded by

$$(4.14)\quad 2\delta C\,|h_n(\tau)| \leq 2\delta Cn\,|w|^{(N+1)/(M-1)}\exp(-\gamma n\,|w|^{N/(M-1)}) \qquad (n = 1, 2, \cdots)$$

if $w$ satisfies (4.7) and if $\delta = o(|w|^{1/(M-1)})$ as $w \to 0$. This restriction on $\delta$ is fulfilled if we take for $\delta$ the number that minimizes the sum of the expression in (4.13) and the term on the right in (4.14). Specifically, let

$$\delta = n^{-1-\varepsilon/2}|w|^{-(N+M-1)/2(M-1)}\exp\left(\frac{n\gamma}{2}|w|^{N/(M-1)}\right).$$

With this $\delta$ (in (4.13) and (4.14)), we find that if $w$ satisfies (4.7) then

$$|b_n(w)| \leq C_\varepsilon n^{-\varepsilon/2}|w|^{(N+3-M)/2(M-1)}\exp\left(\frac{-n\gamma}{2}|w|^{N/(M-1)}\right) \qquad (n = 1, 2, \cdots).$$

This proves the lemma.

We can make corresponding estimates for

$$B_n(w) = \int_0^{\alpha_n} H_n(t)\exp\{in\,G(t)\}\,dt,$$

where $\alpha_n = n^{-1/(N+1)}$, where

$$G(t) = -wt - (\Im\gamma)t^N \tag{4.15}$$

(N a positive integer, $\Im\gamma \leq 0$, with $\Im\gamma = 0$ if $N = 1$), and where

$$H_n(t) = [e^{nv(t)} - 1]\exp\{-n\Re\gamma t^N\} \qquad (n = 1, 2, \cdots)$$

($\Re\gamma > 0$, v a continuous, complex-valued function of bounded variation on $[0, \delta]$, for some $\delta > 0$, and satisfying (4.4)).

LEMMA 4.2. *If* $N \geq 2$ *and* $\Im\gamma < 0$ *in equation* (4.15), *let*

$$|w| \geq C_0\, n^{-(N-1)/(N+1)}, \tag{4.16}$$

*where* $C_0$ *is chosen so that* $|G'(t)| \geq |w|/2$ $(0 \leq t < \alpha_n)$ *whenever* (4.16) *holds. Then, for* $0 < \varepsilon < 1/N$, *there exists a constant* $C_\varepsilon$ *such that*

$$|B_n(w)| \leq C_\varepsilon\, n^{-1-\varepsilon}\, |w|^{-1} \qquad (n = 1, 2, \cdots). \tag{4.17}$$

*Also, if* $N \geq 2$, $\Im\gamma < 0$, *and*

$$|w| \leq C_0\, n^{-(N-1)/(N+1)},$$

*then*

$$|B_n(w)| \leq C_\varepsilon\, |w|^{3/2(N-1)}\, n^{-\varepsilon/2} \exp\left\{\frac{-n}{2}\Re\gamma\, |w|^{N/(N-1)}\right\} \qquad (n = 1, 2, \cdots). \tag{4.18}$$

*Finally, if* $N \geq 1$ *and* $\Im\gamma = 0$, *then, for* $n = 1, 2, \cdots$,

$$|B_n(w)| \leq C_\varepsilon\, n^{-1-\varepsilon}\, |w|^{-1} \quad \textit{if} \quad |w| \geq n^{-1-\varepsilon+1/N+1/(N+1)} \tag{4.19}$$

*and*

$$|B_n(w)| < C\, n^{-1/(N+1)-1/N} \quad \textit{if} \quad |w| < n^{-1-\varepsilon+1/N+1/(N+1)}. \tag{4.20}$$

*Proof.* The proof of (4.17) and (4.18) consists of setting $M = N$ in the proof of Lemma 4.1. For the proof of (4.19), we use integration by parts as in the proof of (4.6). But this time, since $G'(t) = -w$, we see that

$$|B_n(w)| \leq C_\varepsilon\, n^{-1-\varepsilon}\, w^{-1} \qquad (n = 1, 2, \cdots),$$

without any restriction on $|w|$. The restriction on $|w|$ in (4.19) is inserted because the estimate is of no use for smaller $|w|$. The inequality (4.20) is simply the statement

$$|B_n(w)| \leq \alpha_n \sup H_n(t).$$

The lemma is proved.

## 5. PROOFS OF ASSERTIONS (ii) TO (iv)

By combining the results in our lemmas, we can now make estimates of $\|f^n\| = \sum_k |c_{kn}|$ when f satisfies the conditions of cases (ii), (iii), or (iv). We shall begin the section with estimates from above with primary emphasis on case (ii). The estimates from below appear in the second half of the section.

We have seen in Lemma 2.1 that if f satisfies the conditions of any of these theorems, then the inequality

$$\sum_{|k|\geq \mu n} |c_{kn}| \leq C$$

holds for some $\mu > 0$.

Consider the case where $|f(0)| = 1$ (we may take $f(0) = 1$), $|f(t)| < 1$ $(t \neq 0)$, and f satisfies conditions $(D_R)$ and $(C_L)$ at 0. In the other cases, $\|f^n\|$ is estimated in the same way. We are assuming that (1.6) holds to the right of 0 and that an estimate (1.4) with $|t|$ in place of t holds to the left of 0. To distinguish the right from left we put primes on everything appearing on the left of 0. For $|k| \leq \mu n$, we consider the identity

$$(5.1) \qquad 2\pi c_{kn} = \left(\int_{-\pi}^{-\alpha_n'} + \int_{-\alpha_n'}^{0} + \int_{0}^{\alpha_n} + \int_{\alpha_n}^{\pi}\right) f^n(t)\, e^{-ikt} dt.$$

By inequality (2.5), the first and last integrals satisfy the inequality

$$(5.2) \qquad \begin{aligned} |R_n| &= \left|\left(\int_{-\pi}^{-\alpha_n'} + \int_{\alpha_n}^{\pi}\right) f^n(t)\, e^{-ikt} dt\right| \\ &\leq \pi \exp\{-\Re\gamma' n^{1/(N'+1)}\} + \pi \exp\{-\gamma n^{1/(N+1)}\}. \end{aligned}$$

Using (2.6) and (2.7) to express the second and third integrals, we rewrite (5.1) in the form

$$2\pi c_{kn} = A_n'(-w') + B_n'(-w') + a_n(w) + b_n(w) + R_n,$$

where $w' = kn^{-1} - a_1'$ and $w = kn^{-1} - a_1$.

It is sufficiently precise to use the inequality

$$(5.3)' \quad \|f^n\| = \sum_k |c_{kn}| \leq C + \sum_{|k|\leq \mu n} (|A_n'(-w')| + |B_n'(-w')| + |a_n(w)| + |b_n(w)| + R_n).$$

Our estimates of $a_n(w)$ in Section 3 were made under the condition $a_M > 0$. If $a_M < 0$, the estimates hold for $\overline{a_n(-w)}$. Thus, in estimating

$$\sum_{|k|\leq \mu n} |a_n(w)|$$

there is no loss of generality if we assume that $a_M > 0$. The same is true of $A_n(w)$, $B_n(w)$, and $b_n(w)$.

Since $\sum_k |a_n(w)|$ gives the major contribution, we discuss it first. Using Lemmas 3.1, 3.3, 3.5, and 3.6 we find that for some $\beta > 0$, and with the notation

$$\eta_1 = n^{-(M-1)/M}, \qquad \eta_2 = n^{-(M-1)/(N+1)},$$

the inequality

$$\sum_{|k|\leq \mu n} |a_n(w)| \leq Cn \int_{\lambda(n)\eta_1}^{\eta_2} n^{-1/2} w^{-(M-2)/2(M-1)} \exp\{-n\beta w^{N/(M-1)}\}\, dw$$

$$+ C_1 n \int_{\eta_1}^{\lambda(n)\eta_1} w^{1/(M-1)}\, dw + \sum_{|k-na_1|\leq n^{1/M}} C_2 n^{-1/M} + C_3 n \int_{\eta_1}^{2\mu} \frac{dw}{nw}$$

holds for $n = 1, 2, \cdots$.

An easy computation leads to the estimate

$$\sum_{|k|\leq \mu n} |a_n(w)| \leq C n^{(N-M)/2N} + C_1 (\log n)^3 + C_2 + C_3 \log n \quad (n = 1, 2, \cdots). \tag{5.4}$$

A similar computation, based on Lemmas 3.2, 3.4, 3.5, and 3.6 if $N' > 1$, and on Lemma 3.7 if $N' = 1$, gives—with the notation

$$\eta_3 = n^{-(N'-1)/N'}, \quad \eta_4 = n^{-(N'-1)/(N'+1)},$$

—the estimates

$$\sum_{|k|\leq \mu n} |A_n'(-w')| \leq Cn \int_{\lambda_0(n)\eta_3}^{\eta_4} n^{-1/2} w^{-(N'-2)/2(N'-1)} \exp\{-n\beta w^{N'/(N'-1)}\}\, dw$$

(5.5)

$$+ \sum_{|k-na_1'|\leq n^{1/N'}} C_1 n^{-1/N'} + C_2 n \int_{\eta_3}^{2\mu} \frac{dw}{nw} \leq C' \log n \quad (N' > 1),$$

and

$$\sum_{|k|\leq \mu n} |A_n'(-w')| \leq Cn \int_{-2\mu}^{2\mu} \frac{w}{n|w - i\gamma|} \leq C' \log n \quad (N' = 1).$$

Also, it follows easily from Lemmas 4.1 and 4.2 that for every $\varepsilon$ $(0 < \varepsilon < 1/N)$,

$$\sum_{|k|\leq \mu n} |b_n(w)| \leq C_\varepsilon n^{-\varepsilon} \log n + C_\varepsilon' n^{-\left(\frac{\varepsilon}{2}+\frac{M+1}{N}\right)} \quad (n = 2, 3, \cdots), \tag{5.6}$$

and that if $0 < \varepsilon' < 1/N'$, then

$$\sum_{|k|\leq \mu n} |B_n'(-w')| \leq C_{\varepsilon'} n^{-\varepsilon'} \log n + C_{\varepsilon'}' n^{-\left(\frac{\varepsilon'}{2}+\frac{N'+1}{N'}\right)} \quad (n = 2, 3, \cdots). \tag{5.7}$$

Combining inequalities (5.3) to (5.7) we find that

$$\|f^n\| \leq C n^{(N-M)/2N} \quad (n = 1, 2, \cdots).$$

Thus we have proved the second inequality in (1.7) for the case under consideration, and it is clear that a similar argument proves it for the other cases mentioned in assertion (ii).

It is also clear from this argument that if $(C_R)$ and $(C_L)$ hold at 0, then the estimate

$$\|f^n\| \leq C \log n \qquad (n = 2, 3, \cdots)$$

follows from inequalities (5.5) and (5.7). This proves the right half of (1.7) under the conditions of assertion (iii).

Among the estimates from above, we need only prove that $\|f^n\|$ is bounded if $(C_R)$ and $(C_L)$ hold at 0 with $a_1' = a_1$ and $N' = N$. Under these conditions (and with $N > 1$) we have the inequality

$$\|f^n\| = \sum_k |c_{kn}| \leq C + \sum_{|k| \leq \mu n} (|A_n'(-w) + A_n(w)| + |B_n'(-w)| + |B_n(w)| + R_n).$$

Since the $-i/nw$-term of the estimate of $A_n(w)$ is cancelled by the same term of $A_n'(-w)$ in the estimates (3.14), (3.24), and (3.35), we find that the logarithmic term does not appear. In fact, it follows from these same estimates, that with

$$\eta_5 = n^{-(N-1)/N} \qquad \text{and} \qquad \eta_6 = n^{-(N-1)/(N+1)},$$

$$\sum_{|k| \leq \mu n} |A_n'(-w) + A_n(w)|$$

$$\leq Cn \int_{\lambda_0(n)\eta_5}^{\eta_6} n^{-1/2} w^{-(N-2)/2(N-1)} \exp\{-n w^{N/(N-1)}\}\, dw$$

$$+ \sum_{|k - na_1| \leq n^{1/N}} C_1 n^{-1/N} + C_2 n \int_{\eta_5}^{2\mu} n^{-N} w^{-(N+1)}\, dw \leq C_3 \qquad (n = 1, 2, \cdots),$$

and hence $\|f^n\|$ is bounded. The inequalities

$$\sum_{|k| \leq \mu n} |A_n'(-w) + A_n(w)| \leq C \qquad (n = 1, 2, \cdots;\ N = N' = 1)$$

follow easily from (3.36). Thus all the upper estimates in cases (ii) to (iv) are proved.

We now turn to the lower estimates, and again we consider case (ii) first. The most interesting case is where f satisfies conditions $(D_L)$ and $(D_R)$ at 0, and where $N' = N$, $M' = M$, $a_1' = a_1$, and $a_M' = -a_M$. Since $\|\bar{f}\| = \|f\|$, it is no restriction to assume that $a_M > 0$.

We use the estimate

$$\text{(5.8)} \quad \|f^n\| \geq \sum_{k \in I_n} |c_{kn}| \geq \sum_{k \in I_n} |a_n'(-w) + a_n(w)| - \sum_{k \in I_n} (|b_n'(-w)| + |b_n(w)| + R_n),$$

where $k \in I_n$ if and only if

$$w = \frac{k}{n} - a_1 \quad \text{and} \quad \frac{1}{2} n^{-(M-1)/N} \leq w \leq n^{-(M-1)/N}.$$

We have seen in (5.2) and (5.6) that the last sum in (5.8) goes to 0 as $n \to \infty$. It follows from Lemma 3.1 that for $k \in I_n$,

$$|a_n(w)| \geq C n^{-1/2} w^{-(M-2)/2(M-1)} \geq C n^{(M-N-2)/2N}. \tag{5.9}$$

Since $a'_M = -a_M$, it also follows from Lemma 3.1 that

$$a'_n(-w) = \overline{a_n(w)} \{1 + o(1)\} \qquad (n \to \infty)$$

uniformly for $k \in I_n$. Thus

$$|a'_n(-w) + a_n(w)| = 2|a_n(w)| \, |\cos(\arg a_n(w)) + o(1)| \qquad (n \to \infty, \ k \in I_n).$$

Let R be the subset of $I_n$ such that if $k \in R$, then $w = kn^{-1} - a_1$ and $|\arg a_n(w) - j\pi| < 9\pi/20$ for some integer j depending on k. Let S be the complement of R in $I_n$. We shall show that for large n at least one half of the k in $I_n$ are in R. This will enable us to show, using (5.9), that

$$\begin{aligned} \|f^n\| &\geq \sum_{k \in I_n} |c_{kn}| \geq \frac{1}{20} \sum_{k \in R} |a_n(w)| \\ &\geq \frac{n}{2} \cdot \frac{1}{2} n^{-(M-1)/N} C n^{(M-N-2)/2N} = C' n^{(N-M)/2N} \end{aligned} \tag{5.10}$$

for some $C' > 0$, which is what we want to prove.

Suppose that for some fixed n, $k_0$ belongs to S. This means that

$$\left|\arg a_n(w_0) - (2j+1)\frac{\pi}{2}\right| \leq \frac{\pi}{20} \qquad \left(w_0 = \frac{k_0}{n} - a_1\right)$$

for some integer j. If the value n is large enough, then it follows from the estimate

$$\arg a_n(w) = -\beta n w^{M/(M-1)} [1 + O((\log n)^{-3/4})] \qquad (k \in I_n)$$

in Lemma 3.1 that if

$$k_0 + \frac{\pi(M-1)}{5M\gamma} w_0^{-1/(M-1)} \leq k \leq k_0 + \frac{4\pi(M-1)}{5M\gamma} w_0^{-1/(M-1)}, \tag{5.11}$$

then

$$-\frac{19\pi}{20} \leq \arg a_n(w) - (2j+1)\frac{\pi}{2} \leq -\frac{\pi}{20}.$$

Thus if $k \in I_n$ and k satisfies (5.11) with $k_0 \in S$, then $k \in R$. If $k_0 \in S$ and

$$w_0 \leq n^{-(M-1)/N} (1 - C n^{-1+M/N})$$

for some $C > 0$ independent of n, then it will be automatically true that $k \in I_n$ whenever k satisfies (5.11). Also, for any fixed $k_0$ in S there are many k satisfying (5.11), since $w_0 \leq n^{-(M-1)/N}$, that is, $w_0^{-1/(M-1)} \geq n^{1/N}$ for $n = 1, 2, \cdots$ and $k \in I_n$. These estimates show that for large n, intervals of R and S alternate, and that at least one half of the k in $I_n$ are also in R. This shows that (5.10) is valid, and assertion (ii) is proved in the case under consideration.

In the other cases in assertion (ii), the proof is easier, since we have the inequality

$$|c_{kn}| \geq C\,|a_n(w)| \geq C' n^{(M-N-2)/2N} \qquad (k \in I_n,\ C' > 0),$$

and the argument we have given shows that the first inequality in (1.7) holds.

We now turn to the estimates from below in assertions (iii) and (iv). Consider first the case where $(C_R)$ and $(C_L)$ hold at 0 with $N > 1$, $N' \geq 1$, and $a_1' \neq a_1$. Let

$$J_n = \left\{ k\colon na_1 + n^{1/N} \leq k \leq n\left(a_1 + \tfrac{1}{3}\,|a_1 - a_1'|\right)\right\}.$$

Then it follows from the estimates in Sections 3 and 4 that for all large enough values $n$,

$$\|f^n\| \geq \sum_{k\in J_n} |c_{kn}| \geq \sum_{k\in J_n} \frac{1}{4n(k - na_1)} \geq C \log n, \tag{5.12}$$

for some $C > 0$.

In the case where $(C_R)$ and $(C_L)$ hold at 0, with $N > N' \geq 1$ and $a_1' = a_1$, let

$$K_n = \{k\colon n^{1/N} \leq k - na_1 \leq n^{1/N'}\}.$$

It follows from Lemmas 3.2, 3.4, 3.5, 3.6, and 4.2 that if $n$ is large, then

$$\|f^n\| \geq \sum_{k\in K_n} |c_{kn}| \geq \sum_{k\in K_n} \frac{1}{4(k - na_1)} \geq C \log n \qquad (n = 1, 2, \cdots) \tag{5.13}$$

for some $C > 0$. The inequalities (5.12) and (5.13) prove the first inequality in (1.7) in case (iii), when $|f(t)| = 1$ at an isolated point.

The estimates from below in case (iii), when $|f(t)| = 1$ on an interval, are proved in the same way. Only two cases are slightly different. One is the case where

$f(t) = \exp ia_1 t \qquad (0 < t < T < 2\pi)$,

$(C_R)$ holds at $T$, $(C_L)$ holds at 0, with $N' = N > 1$, and $a_1$ is the same for all $t$ $(t \neq 0,\ t \neq T)$.

Here it follows from Lemmas 3.4 and 4.2 that

$$\|f^n\| \geq \sum_{|k-na_1|\leq n^{1/N}} |c_{kn}| \geq \sum_{|k-na_1|\leq n^{1/N}} \left|\frac{i}{k - na_1}\,[\exp\{-i(k-na_1)T\} - 1]\right| - C$$
$$\geq C' \log n - C \qquad (n = 2, 3, \cdots)$$

for some $C' > 0$.

In the other case of interest ($N = N' = 1$ and $f(t) = \exp i\beta t$ for $0 < t < T < 2\pi$), it follows from Lemmas 3.7 and 4.2 that for some $\varepsilon > 0$

$$\|f^n\| \geq \sum_{|k-n\beta|<\varepsilon n} |c_{kn}| \geq \sum_{|k-n\beta|<\varepsilon n} \left|\frac{i}{k - n\beta}\,[\exp\{-i(k - n\beta)T\} - 1]\right| - C$$
$$\geq C' \log n - C \qquad (n = 2, 3, \cdots)$$

for some $C' > 0$. This completes the proof of assertion (iii).

To complete the proof of assertion (iv) we need to show that if $|f(t_0)| = 1$ for some $t_0$, then $\|f^n\| \geq C$ $(C > 0)$. This follows easily from the relations

$$\|f^n\| = \sum |c_{kn} e^{ikt_0}| \geq \left|\sum c_{kn} e^{ikt_0}\right| = |f^n(t_0)| = 1.$$

## 6. THE PROOF OF ASSERTION (i)

Kahane [5, p. 103] used Carlson's inequality [2],

$$\left(\sum_1^\infty a_n\right)^4 < \pi^2 \sum_1^\infty a_n^2 \sum_1^\infty n^2 a_n^2$$

to obtain the estimate from above.

In order to get the estimate from below, we use an argument similar to the proofs of assertions (ii) to (iv). Let (B) hold at $t = 0$. It is no restriction to set $f(0) = 1$ and to require that $a_2 > 0$. Using (1.3), we choose $\alpha > 0$ so small that for $0 \leq t \leq 3\alpha$

$$\log f(t) = ia_1 t + ia_2 t^2 + iv(t)$$

with $v$ real, $v \in C^3$, and

$$(6.1)\quad |v(t)| \leq Ct^3, \quad |v'(t)| \leq C_1 t^2 < a_2 t/4, \quad |v''(t)| \leq C_2 t \leq a_2 \qquad (0 \leq t \leq 3\alpha).$$

Let $p(t)$ be the piecewise linear function

$$p(t) = \begin{cases} t/\alpha & (0 < t < \alpha), \\ 1 & (\alpha \leq t \leq 2\alpha), \\ (3\alpha - t)/\alpha & (2\alpha \leq t \leq 3\alpha), \\ 0 & \text{otherwise}. \end{cases}$$

We shall base our proof on the inequality $\|f^n\| \geq \|pf^n\| / \|p\|$. Let

$$w = \frac{k}{n} - a_1, \qquad \psi(t) = -wt + a_2 t^2 + v(t).$$

We say that $k \in L_n$ if

$$8\alpha a_2/3 \leq w \leq 10\alpha a_2/3.$$

From the conditions (6.1) on $v'$ it follows that if $k \in L_n$, then $\psi'$ is increasing on $[0, 3\alpha]$ and has on this interval precisely one zero $t_0$, with

$$\frac{32\alpha}{27} \leq t_0 \leq \frac{40\alpha}{21}.$$

Let $\delta(n) = n^{-2/5}$ $(n = 1, 2, \cdots)$. If $k \in L_n$ and n is so large that $\alpha \leq t_0 - \delta(n)$ and $t_0 + \delta(n) \leq 2\alpha$, we write

$$(6.2)\qquad \begin{aligned} 2\pi c_{kn} &= \int_{-\pi}^{\pi} p(t) f^n(t) e^{-ikt} dt \\ &= \int_0^{t_0 - \delta(n)} p e^{in\psi} + \int_{t_0 - \delta(n)}^{t_0 + \delta(n)} e^{in\psi} + \int_{t_0 + \delta(n)}^{3\alpha} p e^{in\psi}. \end{aligned}$$

To estimate the first integral on the right of (6.2), we integrate by parts, obtaining the inequality

$$\left| \int_0^{t_0 - \delta(n)} p e^{in\psi} \right| \leq \frac{1}{n\,|\psi'(t_0 - \delta(n))|} + \operatorname*{var}_{[0,\, t_0 - \delta(n)]} \frac{p}{n\psi'}.$$

From (6.1) it follows that $\psi'$ is negative and increasing on this interval, and that $\psi'(t_0 - \delta(n)) \leq -C\delta(n)$ $(C > 0)$. Therefore

$$(6.3)\qquad \left| \int_0^{t_0 - \delta(n)} p e^{in\psi} \right| \leq C n^{-3/5} \qquad (n = 1, 2, \cdots).$$

Likewise,

$$(6.4)\qquad \left| \int_{t_0 + \delta(n)}^{3\alpha} p e^{in\psi} \right| \leq C n^{-3/5} \qquad (n = 1, 2, \cdots).$$

To consider the second integral on the right-hand side of (6.2), we expand $\psi$ in a Taylor series

$$\psi(t) = \psi(t_0) + b(t_0)(t - t_0)^2 + R(t, t_0)(t - t_0)^3.$$

Since $v \in C^3$, we conclude that $\psi \in C^3$ and that $R(t, t_0)$ is bounded for t and $t_0$ in $[\alpha, 2\alpha]$. Using (6.1), we see that if $k \in L_n$, then

$$(6.5)\qquad 0 < a_2/2 \leq b(t_0) \leq 3a_2/2.$$

We write

$$(6.6)\qquad \begin{aligned} \int_{t_0 - \delta(n)}^{t_0 + \delta(n)} e^{in\psi} = e^{in\psi(t_0)} \Bigg[ &\int_{t_0 - \delta(n)}^{t_0 + \delta(n)} e^{inb(t - t_0)^2} dt \\ &+ \int_{t_0 - \delta(n)}^{t_0 + \delta(n)} e^{inb(t - t_0)^2} [e^{inR(t - t_0)^3} - 1]\, dt \Bigg]. \end{aligned}$$

The first integral on the right-hand side of (6.6) can be estimated by integration along the three lines from $t_0 - \delta$ to $t_0 - \delta - i\delta$, then to $t_0 + \delta + i\delta$, and then to $t_0 + \delta$. This gives the estimate

$$\left| \int_{t_0 - \delta(n)}^{t_0 + \delta(n)} e^{inb(t-t_0)^2} dt \right| = \frac{C}{\sqrt{nb}} + O(1/n) \qquad (n \to \infty).$$

From this and (6.5) it follows that if $k \in L_n$ and $n$ is large, then

$$\left| \int_{t_0 - \delta(n)}^{t_0 + \delta(n)} e^{inb(t-t_0)^2} dt \right| \geq C/\sqrt{n} \qquad (C > 0).$$

The last integral in (6.6) satisfies the inequality

$$\left| \int_{t_0 - \delta(n)}^{t_0 + \delta(n)} e^{inb(t-t_0)^2} [e^{inR(t-t_0)^3} - 1] dt \right| \leq 2\delta \max_{|t-t_0| \leq \delta} |e^{inR(t-t_0)^3} - 1|$$

$$\leq C n \delta^4 = C n^{-3/5}.$$

Hence, if $k \in L_n$ and $n$ is large, then

$$\left| \int_{t_0 - \delta(n)}^{t_0 + \delta(n)} e^{in\psi} \right| \geq C/\sqrt{n} \qquad (C > 0), \tag{6.7}$$

If in (6.2) we use the inequalities (6.3), (6.4), and (6.7), we get the estimate

$$|c_{kn}| \geq C/\sqrt{n} \qquad (C > 0,\ k \in L_n,\ n = 1, 2, \cdots).$$

Since there are $C_1 n$ numbers $k$ in $L_n$, we get the required estimate

$$\|f^n\| \geq C \|pf^n\| \geq C' \sum_{k \in L_n} |c_{kn}| \geq C' n.$$

This completes the proof of assertion (i) and the theorem.

## REFERENCES


1. A. Beurling and H. Helson, *Fourier-Stieltjes transforms with bounded powers*, Math. Scand. 1 (1953), 120-126.
2. F. Carlson, *Une inégalité*, Ark. Mat. 25B (1935), no. 1, 1-5.
3. G. W. Hedstrom, *The near-stability of the Lax-Wendroff method*, Numer. Math. 7 (1965), 73-77.
4. J. P. Kahane, *Sur certaines classes de séries de Fourier absolument convergentes*, J. Math. Pures Appl. (9) 35 (1956), 249-259.
5. Y. Katznelson, *Sur le calcul symbolique dans quelques algèbres de Banach*, Ann. Sci. École Norm. Sup. (3) 76 (1959), 83-123.
6. Z. L. Leĭbenzon, *On the ring of functions with absolutely convergent Fourier series*, Uspehi Matem. Nauk (N.S.) 9 (1954) no. 3 (61), 157-162. (Russian.)
7. F. Riesz and B. Sz.-Nagy, *Functional analysis*, Ungar, New York, 1955.

8. S. I. Serdjukova, *A study of stability in* C *of explicit difference schemes with constant real coefficients which are stable in* $\ell_2$, Ž. Vyčisl. Mat. i Mat. Fiz. 3 (1963), 365-370. (Russian.)

9. H. J. Stetter, *Maximum bounds for the solutions of initial value problems for partial difference equations*, Numer. Math. 5 (1963), 399-424.

10. G. Strang, *Polynomial approximation of Bernstein type*, Trans. Amer. Math. Soc. 105 (1962), 525-535.

11. V. Thomée, *Stability of difference schemes in the maximum-norm*, J. Differential Equations 1 (1965), 273-292.

*Remark.* The recent paper of S. I. Serdjukova, *On the stability in* C *of linear difference schemes with constant real coefficients*, Ž. Vyčisl. Mat. i Mat. Fiz. 6 (1966), 477-486, (Russian), contains special results similar to ours.

The University of Michigan