PROJECTIVE AND INVERSIVE MODELS FOR FINITE HYPERBOLIC PLANES

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1. INTRODUCTION

A *finite hyperbolic plane* $\mathscr{H}$ is a collection of *points*, together with certain distinguished subsets called *lines*, satisfying the following conditions (see [6], [7]):

H.1 Two distinct points are contained in exactly one line.

H.2 If P is a point not contained in line ℓ, then there exist at least two lines containing P and not meeting ℓ.

H.3 If a subset $\mathscr{S}$ of the points of $\mathscr{H}$ contains three points not on a line and contains all points on lines through pairs of distinct points of $\mathscr{S}$, then $\mathscr{S}$ contains all the points of $\mathscr{H}$.

A number of examples of such planes have recently been given [1], [6], [7], [8], [9]. In particular, it is shown in [1] that, with suitable identification of inverse points, the circles *tangent* to a given circle in the classical inversive plane over $GF(2^t)$ yield a model of a finite hyperbolic plane. One purpose of the present note is to extend this result to $GF(p^t)$ (p an odd prime) with "orthogonal" replacing "tangent". The recent results of Dembowski and Hughes [5] show that the inversive planes over $GF(p^t)$ (p odd) are exactly the abstract inversive planes of odd order with orthogonality. Hence we derive our model in the context of abstract inversive planes with orthogonality. Since abstract inversive planes of even order necessarily have an orthogonality relation, namely tangency, the model derived in the present paper includes as a special case the hyperbolic planes described in [1] and [2].

For the case of odd order, an isomorphism is established, in Section 3, between the present inversive model of a hyperbolic plane and Ostrom's projective model. In Section 4 we include the special calculations needed for planes of order at most 7.

2. THE MODEL

An abstract inversive plane $\mathscr{I}$ is a collection of *points* and certain distinguished subsets of points called *circles*, satisfying the following three conditions:

I.1 Three distinct points are contained in exactly one circle.

I.2 If c is a circle containing point Q but not point P, then there exists exactly one circle d containing P and tangent to c at Q. (By definition, two point sets are *tangent* if and only if they have exactly one point in common.)

I.3 There are at least two circles. Every circle contains at least three distinct points.

Received April 5, 1965.
This research was supported by a National Science Foundation Grant.

It can be verified by simple counting methods [4] that if one circle contains exactly $n+1$ points, then every circle contains exactly $n+1$ points, every pair of points is contained in exactly $n+1$ circles, and there are exactly n^2+1 points in $\mathscr{I}$. The number n is called the *order* of $\mathscr{I}$.

An *inversive plane with orthogonality* is an inversive plane $\mathscr{I}$ together with a relation $c \perp d$ between the circles of $\mathscr{I}$ satisfying [5] the following requirements.

O.1 If $c \perp d$, then $d \perp c$.

O.2 If P and Q are different points and if c is a circle containing Q, then there exists exactly one circle d such that P and Q are in d, and $c \perp d$.

O.3 If a and b are different circles, each containing two distinct points P and P^*, and if $a \perp c$ and $b \perp c$, then $x \perp c$ for all circles x containing P and P^*.

Dembowski and Hughes [5] have shown that in case n is even, the only possible orthogonality relation $c \perp d$ is the following: $c = d$ or c is tangent to d. For the proof of the following result, see [5, p. 176].

LEMMA 1. *The circles containing a given point* P *and orthogonal to a given circle* c *(not containing* P*) are exactly the* $n+1$ *circles through* P *and some other point* $P^* \notin c$.

For any fixed circle c in an inversive plane $\mathscr{I}$ of order n ($n > 7$), we now construct a hyperbolic plane $\mathscr{H}_c(\mathscr{I})$. By Lemma 1, each point $P \notin c$ determines a unique point P^* contained in each of the $n+1$ circles orthogonal to c and containing P. The *points* of $\mathscr{H}_c(\mathscr{I})$ are the pairs $\overline{P} = \{P, P^*\}$ ($P \notin c$). A *line* $\overline{\ell}$ of $\mathscr{H}_c(\mathscr{I})$ is the set of all points $\overline{P} = \{P, P^*\}$ for which P lies on the circle ℓ orthogonal to c. The verification of axioms H.1 to H.3 follows:

H.1. Let $\overline{P} = \{P, P^*\}$ and $\overline{Q} = \{Q, Q^*\}$ be two distinct points. The $n+1$ circles through P and Q include all points of $\mathscr{I}$, in particular P^*. The unique circle ℓ containing P, P^*, and Q is orthogonal to c, by Lemma 1. Hence ℓ contains Q^*, and $\overline{\ell}$ is the required line determined by $\overline{P}$ and $\overline{Q}$.

H.2. Since each circle $\ell \perp c$ contains exactly $n+1$ points, the corresponding line $\overline{\ell}$ of $\mathscr{H}_c(\mathscr{I})$ contains (by identification in pairs of those points of ℓ that are not on c) at most $(n+1)/2$ points of $\mathscr{H}_c(\mathscr{I})$. However, at each point $\overline{P} \notin \overline{\ell}$ there are exactly $n+1$ lines, by Lemma 1. Hence at least $(n+1) - (n+1)/2 \geq 2$ of them fail to meet $\overline{\ell}$, as required.

To prove H.3, it is convenient to establish the following extension of an observation of Szamkołowicz [10, Theorem 3.1]. In an incidence structure $\mathscr{G}$ satisfying H.1 and H.2, we denote by k the minimum number of points on a line, and by $m+1$ the maximum number of lines containing any point.

LEMMA 2. *If* H.1 *and* H.2 *are satisfied in an incidence structure* $\mathscr{G}$, *and* $k(k-1) > m$, *then* H.3 *is satisfied in* $\mathscr{G}$.

Proof. Let $\overline{P}, \overline{Q}, \overline{R}$ be three noncollinear points of a subset $\mathscr{S}$ of points of $\mathscr{G}$ having the property that if any line contains two distinct points of $\mathscr{S}$, then all its points are in $\mathscr{S}$. $\mathscr{S}$ contains at least the k points of line $\overline{PQ}$, and hence the $k(k-1)+1$ distinct points on lines joining $\overline{R}$ to points of $\overline{PQ}$. Now each point $\overline{T}$ of $\mathscr{G}$ is on at most $m+1$ lines. Hence if $k(k-1)+1 > m+1$, then at least one of the lines containing $\overline{T}$ must contain (at least) two points of $\mathscr{S}$. Hence all the points of this line, in particular, $\overline{T}$, are in $\mathscr{S}$, as required.

(We remark that in a weak sense, Lemma 2 is a best possible result. That is, there exist incidence structures that satisfy H.1, H.2, and the condition $k(k - 1) = m$, but fail to satisfy H.3. Indeed, consider a projective three-dimensional space with k points on each line. In this space there are $(k - 1)^2 + (k - 1) + 1 = k(k - 1) + 1$ lines containing each point; therefore $m = k(k - 1)$. It is easy to see that H.1 and H.2 hold, but not H.3.)

Proof of H.3. For $\mathscr{H}_c(\mathscr{I})$ we know that $m = n$. If n is odd, then $k = (n - 1)/2$. Hence, by Lemma 2, we need only ascertain that

$$\frac{1}{2}(n - 1)\left[\frac{1}{2}(n - 1) - 1\right] > n,$$

which is the case whenever $n > 7$. If n is even, then $k = n/2$, and the condition $\frac{n}{2}\left(\frac{n}{2} - 1\right) > n$ holds whenever $n > 6$. (In fact, H.3 also holds if $n = 2$, 3, or 7. The cases $n \leq 7$ are treated in Section 4.)

3. ISOMORPHISM OF INVERSIVE AND PROJECTIVE MODELS

In this section, n is always odd. In a projective plane $\mathscr{P}_2$ of odd order n (that is, with $n + 1$ points on each line), an *oval* consists of $n + 1$ points no three of which are collinear. The *interior points* of an oval q are the points of $\mathscr{P}_2$ contained in no tangent line of q. Ostrom [7] has shown that if $n > 7$, the interior points of an oval q in $\mathscr{P}_2$ constitute the points of a finite hyperbolic plane $\mathscr{H}_q(\mathscr{P}_2)$ whose lines are the intersections of the lines of $\mathscr{P}_2$ (not tangent to q) with the set of interior points of q. Thus every line of $\mathscr{H}_q(\mathscr{P}_2)$ contains exactly $(n - 1)/2$ or $(n + 1)/2$ points, depending on whether the corresponding line of $\mathscr{P}_2$ meets q (in two points) or fails to meet q. This suggests an isomorphism (which we now establish) between the inversive model of Section 2 and Ostrom's projective model.

Let $\mathscr{Q}$ be a quadric in a projective three-dimensional space $\mathscr{P}_3$ of odd order n. Let P^ϕ and π^ϕ be the plane and the point corresponding to the point P and the plane π, respectively, in the polarity ϕ induced by $\mathscr{Q}$. A plane section a of $\mathscr{Q}$ is *nontrivial* if it consists of more than one point. Let π_a denote the plane defining such a plane section a, so that $a = \pi_a \cap \mathscr{Q}$. Dembowski and Hughes [5] have shown that any inversive plane $\mathscr{I}$ (of odd order n) with orthogonality is isomorphic to an inversive plane $\mathscr{I}(\mathscr{Q})$ consisting of the points and (nontrivial) plane sections of $\mathscr{Q}$. Moreover, under this isomorphism, nontrivial plane sections a and b of $\mathscr{Q}$ correspond to orthogonal circles in $\mathscr{I}$ if and only if $\pi_a^\phi \in \pi_b$. That is, an orthogonality relation satisfying O.1 to O.3 is defined in $\mathscr{I}(\mathscr{Q})$ by the condition

$$(0) \qquad a \perp b \text{ if and only if } \pi_a^\phi \in \pi_b.$$

Under this isomorphism, each hyperbolic plane derived from a circle of $\mathscr{I}$ corresponds (isomorphically) to a hyperbolic plane $\mathscr{H}_c(\mathscr{Q})$ derived from a plane section c of $\mathscr{Q}$. Let $N = \pi_c^\phi$ be the pole of the plane π_c of the fixed plane section c. The carriers of the *lines* of $\mathscr{H}_c(\mathscr{Q})$ are the plane sections of $\mathscr{Q}$ orthogonal to c. By relation (0), they all lie in planes through N. Of these planes, those containing a given point $P \notin c$ of $\mathscr{Q}$ contain the entire line PN. Hence they also contain the other point of intersection of PN with $\mathscr{Q}$. This point is the P^* of Lemma 1. That is, the *points* of $\mathscr{H}_c(\mathscr{Q})$ are the pairs $\overline{P} = \{P, P^*\}$ that are the complete intersections with $\mathscr{Q}$ of nontangent lines to $\mathscr{Q}$ from N.

Now let X' denote the image in the fixed plane π_c of the point $X \in \mathcal{Q}$ under projection from N; that is, let $X' = NX \cap \pi_c$. Denote by a' the set of images of points X of a nontrivial plane section a of $\mathcal{Q}$. Clearly, c' = c is an oval in π_c. If d is orthogonal to c, then d' is contained in a line of π_c, since $N \in \pi_d$. In fact, if $P \in d$ (and $P \notin c$), then $P' = P^{*\prime}$ is an interior point of c' in π_c. That is, there exist no tangents to c' that contain P', for all the tangent lines to c' (in π_c) are images of planes *tangent* to $\mathcal{Q}$ at points of c, and hence can contain no points P' corresponding to points of $\mathcal{Q}$ not on c.

Under this projection, both the components P and P* of a point $\overline{P}$ of $\mathcal{H}_c(\mathcal{Q})$ project onto the same interior point P' of c in π_c. Hence a one-to-one correspondence between points $\overline{P}$ of $\mathcal{H}_c(\mathcal{Q})$ and interior points of c in π_c is defined by $\tau: \overline{P} = \{P, P^*\} \to P'$. The image $\tau\overline{l}$ of any line $\overline{l}$ of $\mathcal{H}_c(\mathcal{Q})$ is then the intersection of a line of π_c with the set of interior points of c in π_c. Incidences are obviously preserved under τ. Thus τ gives the desired isomorphism between $\mathcal{H}_c(\mathcal{Q})$ and one of Ostrom's models of a finite hyperbolic plane in the (necessarily Desarguesian) projective plane π_c.

4. THE CASES $n \leq 7$

For completeness we include a proof that, in fact, condition H.3 fails if n = 4 or 5 and holds for all other admissible (that is, prime power) values of $n \leq 7$, namely for n = 2, 3, or 7.

For n = 2 there is only one point in $\mathcal{H}_c(\mathcal{I})$, and for n = 3 there are only three points. Hence H.3 is satisfied trivially in these cases.

For n = 4 there are exactly six points and 15 lines. Each line contains exactly two of the points. (That is, the points and lines are the vertices and edges of the complete graph on six vertices.) Any set $\mathcal{S}$ consisting of three noncollinear points has the property described in H.3. But $\mathcal{H}_c(\mathcal{I})$ contains six points. Hence $\mathcal{S} \neq \mathcal{H}_c(\mathcal{I})$, and condition H.3 fails to hold.

For n = 5 there are ten points. It is possible to choose a set $\mathcal{S}$ of four points P, Q, R, S with the properties that

P, Q, and R are noncollinear,

S lies on line QR,

the lines PQ, PS, and PR each contain only two points, and

the line QR contains only three points.

Thus $\mathcal{S}$ has the property described in H.3, but $\mathcal{S} \neq \mathcal{H}_c(\mathcal{I})$. Hence H.3 fails to hold. (Explicitly, consider Ostrom's model, with c the conic $x^2 + 2y^2 + 4z^2 = 0$ over GF(5). The interior points of c are those points (x_0, y_0, z_0) for which $x_0^2 + 2y_0^2 + 4z_0^2$ is a nonzero square in GF(5). Then the points

$$P = (0, 1, 1), \quad Q = (3, 3, 1), \quad R = (2, 3, 1), \quad S = (1, 0, 0)$$

satisfy the conditions just described.)

For n = 7 we shall show that any set $\mathcal{S}$ satisfying the conditions of H.3 contains at least nine points. Since each point T of $\mathcal{H}_c(\mathcal{I})$ is on exactly eight lines, this shows that at least one line containing T contains at least two points of $\mathcal{S}$. Hence (by the definition of $\mathcal{S}$) $T \in \mathcal{S}$, and H.3 is satisfied.

Let P, Q, R be three noncollinear points of $\mathscr{S}$. Then, since there are at least three points on each line, $\mathscr{S}$ contains at least three other points, say S, U, V, on lines QR, PR, and PQ, respectively. The line PS contains another point, say W. It can be readily verified that if none of the lines VR, UQ, or UV contains a new point of $\mathscr{H}_c(\mathscr{S})$, then these seven points of $\mathscr{S}$, together with the seven lines determined by them, constitute a Fano configuration. By the isomorphism established in Section 3, this implies the existence of a Fano configuration in the Desarguesian projective plane coordinatized by GF(7), which is impossible.

Hence at least one of the lines VR, UQ, or UV contains a new point X of $\mathscr{H}_c(\mathscr{S})$. If $X \notin PS$, then line PX contains a third point Y that is different from the eight points already enumerated. Hence $\mathscr{S}$ contains at least nine points, as required.

If $X \in PS$, there are three cases:

$$\text{(i) } X \in VR, \qquad \text{(ii) } X \in UQ, \qquad \text{(iii) } X \in UV.$$

In case (i), either WQ or XQ contains a new point, as required. Likewise, in case (ii) either WR or XR contains such a new point. In case (iii), line RX contains a new point, as required. This completes the proof that $\mathscr{S}$ always contains at least nine points and hence that H.3 holds.

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