SOME IN-BETWEEN THEOREMS FOR DARBOUX FUNCTIONS

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Suppose $\mathscr{C}$ is a class of real-valued functions on a real interval I, closed under scalar multiplication but not closed under addition. For f, g $\in \mathscr{C}$, we say that $g < f$ provided $g(x) < f(x)$ for all $x \in I$, and we raise the question whether the relation $g < f$ implies the existence of an element h in $\mathscr{C}$ such that $g < h < f$. Of course, if $\mathscr{C}$ were additive, then the average of f and g would give such an h.

We shall attack this question for the following specific classes: $\mathscr{D}$, the class of Darboux functions; $\mathscr{U}$, the uniform closure of $\mathscr{D}$; $\mathscr{D}\mathscr{B}_\alpha$, the class of Darboux functions that are Borel measurable of class α; and $\mathscr{U}\mathscr{B}_\alpha$. All these classes have an interesting structure, yet they are badly nonadditive. For example, each function is the sum of two $\mathscr{D}$-functions; and $f + g \notin \mathscr{D}\mathscr{B}_1$ when f, g are the following $\mathscr{D}\mathscr{B}_1$-functions on the line:

$$f(x) = \sin\frac{1}{x} \text{ for } x \neq 0 \quad \text{and} \quad f(0) = 0,$$

$$g(x) = -f(x) \text{ for } x \neq 0 \quad \text{and} \quad g(0) = 1.$$

(For the foregoing facts about Darboux functions and for other information used in the sequel, we refer the reader to the expository paper [2] by Bruckner and Ceder.)

It turns out that there exists a pair of comparable $\mathscr{D}\mathscr{B}_2$-functions that admit no $\mathscr{D}$-function between them (Example 1). Nevertheless, we find a reasonable sufficient condition on a pair of comparable functions to admit a $\mathscr{D}$-function between them (Theorem 1), a condition that is satisfied, for example, by every pair of comparable $\mathscr{D}\mathscr{B}_1$-functions. Actually we prove that a $\mathscr{D}\mathscr{B}_2$-function can be inserted between two comparable $\mathscr{D}\mathscr{B}_1$-functions (Theorem 3). Whether this inserted function may be chosen to belong to $\mathscr{D}\mathscr{B}_1$ is an interesting unsolved problem. With reference to $\mathscr{U}$-functions, our results are more complete, in particular, we prove that any two comparable $\mathscr{U}$-functions admit an intermediate $\mathscr{U}$-function (Theorem 2).

Except where it is otherwise specified, all the functions considered in the sequel will be real-valued functions defined on some real interval I. For convenience, open and closed intervals will be denoted by (a, b) and [a, b], respectively, whether or not $a < b$. We think of an ordinal as the union of all smaller ordinals. Moreover, we shall consider cardinals as ordinals that are not equipollent with smaller ordinals. For any set A, $|A|$ denotes the cardinality of A. The cardinality of the reals is denoted by $\mathfrak{c}$. When $g < f$ and a, b $\in$ I, we define M(a, b) to be the open interval determined by $\min\{g(a), g(b)\}$ and $\max\{f(a), f(b)\}$.

A function f on I is called a *Darboux function* if it takes connected sets onto connected sets. In Bruckner, Ceder, and Weiss [3] the uniform closure $\mathscr{U}$ of $\mathscr{D}$ was characterized as the class of functions f such that $f([a, b] - C)$ is dense in $(f(a), f(b))$ whenever a, b $\in$ I with $f(a) \neq f(b)$ and $|C| < \mathfrak{c}$. Moreover, $\mathscr{U}\mathscr{B}_\alpha$ is precisely the uniform closure of $\mathscr{D}\mathscr{B}_\alpha$, as was shown in [3] and in Ceder and Weiss [5]. For facts about the function classes $\mathscr{B}_\alpha$, see Kuratowski [6].

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We begin by giving a sufficient (and obviously not necessary) condition that two functions admit an intermediate $\mathcal{D}$-function.

THEOREM 1. *Let* $g < f$, *and suppose that* $|\{z \in [a, b]: g(z) < \lambda < f(z)\}| = \mathfrak{c}$ *whenever* $a, b \in I$ *and* $\lambda \in M(a, b)$. *Then there exists an* $h \in \mathcal{D}$ *such that* $g < h < f$. *Moreover, for each* λ *either* $h^{-1}(\lambda) = \Lambda$ *or* $|h^{-1}(\lambda)| = \mathfrak{c}$.

Proof. First well-order $I \times \mathfrak{c}$ by the ordinal $\mathfrak{c}$ so that $I \times \mathfrak{c} = \{z_\xi\}_{\xi < \mathfrak{c}}$, and denote each z_ξ by $\langle x_\xi, \alpha_\xi \rangle$. Also, well-order each open interval J by $\mathfrak{c} - \{0\}$ so that $J = \{r_{J,\xi}\}_{0 < \xi < \mathfrak{c}}$.

Now we proceed to define h inductively as follows:

For the initial step, we can assume without loss of generality that $x_0 \neq x_1$. Next, find two distinct numbers a_0 and a_1 such that

$$g(x_0) < a_0 < f(x_0) \quad \text{and} \quad g(x_1) < a_1 < f(x_1).$$

Then pick $y \in (x_0, x_1)$ so that $g(y) < r_{J,1} < f(y)$, where $J = (a_0, a_1)$. Now put

$$h_1(x_0) = a_0, \quad h_1(x_1) = a_1, \quad h_1(y) = r_{J,1}, \quad D_1 = \{x_0, x_1, y\}.$$

Next, suppose that for each $\beta < \xi$ we have defined a function h_β with domain D_β such that

(1) $x_\alpha \in D_\beta$ whenever $\alpha \leq \beta$,

(2) $|D_\beta| < \mathfrak{c}$,

(3) for each $\alpha < \beta$ with $x_\alpha \neq x_\beta$ and $J = (h_\beta(x_\alpha), h_\beta(x_\beta)) \neq \Lambda$ the relation $|[x_\alpha, x_\beta] \cap h_\beta^{-1}(r_{J,\delta})| = |\beta|$ holds for all $\delta < \alpha$.

Then we proceed to define h_ξ. If $x \in \bigcup_{\beta < \xi} D_\beta$, we define $h'_\xi(x) = h_\delta(x)$, where δ is the first ordinal such that $x \in D_\delta$. If $x_\xi \notin \bigcup_{\beta < \xi} D_\beta$, define $h'_\xi(x_\xi)$ to be any point in $(g(x_\xi), f(x_\xi))$. In order to extend h'_ξ further, we need the following lemma.

LEMMA A. *Suppose* $K \subset I$, $|K| < \mathfrak{c}$, *and* k *is a function on* K *such that* $g < k < f$ *on* K. *If* $b \in K$, $\xi < \mathfrak{c}$, *and* y *is a function from* ξ *onto* $K - k^{-1}(k(b))$, *then* k *can be extended to a function* e *that has domain* E *with* $|E| < \mathfrak{c}$, *and such that, for each* $\delta < \xi$ *and each* η *with* $y(\eta) \in K - k^{-1}(k(b))$, *the equation*

$$|[y(\eta), b] \cap e^{-1}(r_{J,\delta})| = |\xi|$$

holds, where $J = (k(y(\eta)), k(b))$.

Proof of Lemma A. Well-order the set $\{\langle \beta, \alpha \rangle : \alpha \leq \beta < \xi\}$ by the ordinal ξ as $\{z_\mu\}_{\mu < \xi}$ with $z_\mu = \langle \beta_\mu, \alpha_\mu \rangle$. By induction on ξ, we shall now construct a collection of sets $\{A_\mu\}_{\mu < \xi}$.

Choose A_0 to be a subset of $\{x \in [y(\beta_0), b] - K: g(x) < r_{J,\alpha_0} < f(x)\}$ with cardinality $|\xi|$, where $J = (k(y(\beta_0)), k(b))$. Clearly, this is possible, by the hypothesis of the theorem.

Now suppose that for each $\sigma < \mu$ we have defined a set A_σ such that

$$A_\sigma \subseteq \{x \in [y(\beta_\sigma), b] - K: g(x) < r_{J,\alpha_\sigma} < f(x)\} - \bigcup_{\tau < \sigma} A_\tau$$

and $|A_\sigma| = |\xi|$, where $J = (k(y(\beta_\sigma)), k(b))$.

To define A_μ, we take any set of cardinality ξ contained in

$$\{x \in [y(\beta_\mu), b] - K: g(x) < r_{J,\alpha_\mu} < f(x)\} - \bigcup_{\sigma<\mu} A_\sigma .$$

This then completes the definition of the disjoint sets A_μ $(\mu < \xi)$.

Now define $e = k$ on K, and for $x \in A_\mu$ and $J = (k(y(\beta_\mu)), k(b))$, define $e(x) = r_{J,\alpha_\mu}$. Finally, put $E = K \cup \bigcup_{\mu<\xi} A_\mu$. Clearly $|E| < \mathfrak{c}$. Now let

$$y(\eta) \in K - k^{-1}(k(b)) \quad \text{and} \quad \delta < \xi \quad \text{and} \quad r_{J,\delta} \in J = (k(y(\eta)), k(b)).$$

Then put $\sigma = \langle \eta, \delta \rangle$. By construction, $e(A_\sigma) = r_{J,\delta}$, $A_\sigma \subset [y(\eta), b]$, and $|A_\sigma| = |\xi|$. Therefore e possesses the desired properties, and Lemma A is proved.

Now we apply Lemma A, taking the function k to be h'_ξ, taking $b = x_\xi$, and taking y to be a relabelling of the points $\{x_\alpha\}_{\alpha<\xi}$ that lie in $K - k^{-1}(k(b))$. Then we obtain a function $e = h_\xi$ having domain D_ξ and satisfying conditions (1) and (2). Furthermore, for each $\mu < \xi$ with $x_\mu \neq x_\xi$ and with $J = (h_\xi(x_\xi), h_\xi(x_\mu)) \neq \Lambda$, the equation

$$|[x_\xi, x_\mu] \cap h_\xi^{-1}(r_{J,S})| = |\xi|$$

is satisfied for all $\delta < \xi$. But this is precisely condition (3).

Having now completed the construction of the functions $\{h_\xi\}_{0<\xi<\mathfrak{c}}$, we put $h = \bigcup_{0<\xi<\mathfrak{c}} h_\xi$. Clearly, h is a function having domain I and satisfying the condition $g < h < f$. To complete the proof, it remains to show that $h \in \mathscr{D}$. Suppose $h(a) \neq h(b)$ and $\lambda \in J = (h(a), h(b))$. Then $\lambda = r_{J,\alpha}$ for some $\alpha < \mathfrak{c}$. Obviously,

$$|\{\xi \geq \alpha: x_\xi = a\}| = |\{\xi \geq \alpha: x_\xi = b\}| = \mathfrak{c}.$$

Hence we can choose μ and η so that $\mu > \eta > \alpha$ and $x_\mu = a$ and $x_\eta = b$. Then $|[a, b] \cap h^{-1}(\lambda)| = |\mu|$, by the construction of h. But since there are $\mathfrak{c}$ such choices for μ, it follows that $|[a, b] \cap h^{-1}(\lambda)| = \mathfrak{c}$. Hence, $h \in \mathscr{D}$, and moreover, $h^{-1}(\lambda)$ is empty or has cardinality $\mathfrak{c}$ for each λ. This completes the proof of Theorem 1.

Although some pairs of comparable $\mathscr{D}$-functions admit no intermediate $\mathscr{D}$-function (see Example 1), every pair admits an intermediate $\mathscr{U}$-function. In fact, using the method of proof of Theorem 1, we can insert a $\mathscr{U}$-function between any two comparable $\mathscr{U}$-functions. To show this, we need the following lemma, whose conclusion is a weakened version of the hypothesis of Theorem 1.

LEMMA 1. *Let* $g < f$ *and* $f, g \in \mathscr{U}$, *and let* D *be any countable dense subset of the reals. Then for each* $a, b \in I$ *the set*

$$\{\lambda \in M(a, b): |\{z \in [a, b]: g(z) < \lambda < f(z)\}| = \mathfrak{c}\}$$

is dense in $D \cap M(a, b)$.

Proof. Let $J = (d, e)$ be any open interval in $M(a, b)$. Let

$$Z = \{z \in [a, b]: (g(z), f(z)) \cap J \neq \Lambda\}.$$

Then $|Z| = \mathfrak{c}$. To show this, suppose that $f(z) < e$ for some $z \in [a, b]$. Then, since $f \in \mathscr{U}$, it follows that f maps $\mathfrak{c}$ points in (a, b) into J, whence $|Z| = \mathfrak{c}$. Similarly, if $g(z) > d$ for some $z \in [a, b]$, then $|Z| = \mathfrak{c}$. On the other hand, if there were no $z \in [a, b]$ for which $g(z) > d$ or $f(z) < e$, then it would follow that $J \subseteq (g(z), f(z))$ for all $z \in [a, b]$, in which case $|Z| = \mathfrak{c}$. Hence, $|Z| = \mathfrak{c}$ in all cases.

For each $z \in Z$, the open interval $(g(z), f(z))$ contains a member of $D \cap J$. Hence, since $|Z| = \mathfrak{c}$ and $|D| = \aleph_0$, it follows that there is some $r \in D \cap J$ for which $|\{z \in Z: r \in (g(z), f(z))\}| = \mathfrak{c}$; this concludes the proof.

THEOREM 2. *Let* $g < f$ *and* $f, g \in \mathscr{U}$, *and let* D *be any countable dense subset of the reals. Then there exists a* $\mathscr{U}$*-function* h *from* I *into* D *such that* $g < h < f$.

Proof. Our proof is a modification of the proof of Theorem 1, with Lemma 1 "replacing" the hypothesis of Theorem 1. For $a, b \in I$, put

$$R_{a,b} = \{r \in D: |\{z \in (a, b): g(z) < r < f(z)\}| = \mathfrak{c}\}.$$

By Lemma 1, $R_{a,b}$ is dense in M(a, b). We enumerate each $R_{a,b}$ as $\{r_{a,b,n}\}_{n=1}^{\infty}$.

The construction of h_1 is as before, except that we use $r_{a,b,1}$ instead of $r_{J,1}$. Now suppose that for each $\beta < \xi$ we have defined a function h_β with domain D_β such that conditions (1) and (2) are satisfied, together with (3)': for each $\alpha < \beta$ with $x_\alpha \neq x_\beta$ and $J = (h_\beta(x_\alpha), h_\beta(x_\beta)) \neq \Lambda$,

$$|[x_\alpha, x_\beta] \cap h_\beta^{-1}(r)| = |\alpha| \quad \text{for all } r \in J \cap \{r_{a,b,n}: n < \alpha\}.$$

We modify Lemma A so that its conclusion is

$$|[y(\eta), b] \cap e^{-1}(r)| = |\xi| \quad \text{for all } r \in (k(y(\eta)), k(b)) \cap \{r_{y(\eta),b,n}: n < \xi\}.$$

This can be done without difficulty if one begins by well-ordering not

$$\{\langle \beta, \alpha \rangle: \alpha \leq \beta < \xi\},$$

but $\{\langle \beta, s \rangle\}$, where s is restricted to the appropriate countable subset of D.

As before, we define h'_ξ and extend it to h_ξ by the modified version of Lemma A. With h defined as before, it merely remains to show $h \in \mathscr{U}$. To this end, suppose $h(a) \neq h(b)$ and $s \in R_{a,b}$. Then $s = r_{a,b,m}$ for some m. Now choose μ and η so that $\mu > \eta > \omega_0$ with $x_\mu = a$ and $x_\eta = b$. Then, by construction,

$$|[a, b] \cap h_\mu^{-1}(r)| = |\mu| \quad \text{for all } r \in (a, b) \cap \{r_{a,b,n}: n < \eta\}.$$

In particular, $|[a, b] \cap h^{-1}(s)| = |\mu|$. But since $|\{\mu: x_\mu = a\}| = \mathfrak{c}$, we see that $|[a, b] \cap h^{-1}(s)| = \mathfrak{c}$. Hence, $h \in \mathscr{U}$, and Theorem 2 is proved.

Before showing how to insert a $\mathscr{DB}_2$-function between two comparable $\mathscr{DB}_1$-functions, we need four lemmas. We suspect that the first of them is known in some form; but since we have not been able to find a reference for it, we outline its proof.

LEMMA 2. *Let* B *be an uncountable Borel subset of* I, *and let* f *and* g *be Borel measurable functions on* I. *Then there exists a perfect subset* P *of* B *such that* $f | P$ *and* $g | P$ *are continuous.*

Proof. Since B is an uncountable Borel set, it contains a perfect subset Q, which in turn contains a homeomorph C of the Cantor set. But on C there exists a nontrivial Lebesgue-Stieltjes measure λ (Besicovitch [1]). Therefore, we can apply Lusin's theorem to get closed sets F_1 and F_2 such that $\lambda(C - F_i) < 1/2$ and $f \mid F_1$ and $g \mid F_2$ are continuous. But $F_1 \cap F_2$, having positive measure, is a closed set of cardinality $\mathfrak{c}$. Hence, it contains a perfect subset P; this completes the proof. (An alternate proof may be devised by means of the results on p. 306 in [6].)

The next lemma asserts, in part, that two comparable $\mathscr{D}\mathscr{B}_1$-functions satisfy the hypothesis of Theorem 1, and that hence they admit a $\mathscr{D}$-function between them.

LEMMA 3. *Let* $g < f$ *and* $f, g \in \mathscr{D}\mathscr{B}_1$. *Then* $\{z \in [a, b]: g(z) < \lambda < f(z)\}$ *is an uncountable* F_σ*-set, for all* $a, b \in I$ *and each* $\lambda \in M(a, b)$.

Proof. Let $a, b \in I$ and $\lambda \in M(a, b)$. Let C be the F_σ-set

$$\{z \in [a, b]: g(z) < \lambda < f(z)\}.$$

Put $A = [\lambda \leq g]$ and $B = [f \leq \lambda]$. If either of A or B is empty, then $|C| = \mathfrak{c}$, since $g, f \in \mathscr{D}$. We therefore assume that $A \neq \Lambda \neq B$ and that $|C| < \mathfrak{c}$. Then $A \cap B = \Lambda$, $A \cup B \cup C = I$, and $|C| \leq \aleph_0$. Since $f, g \in \mathscr{D}\mathscr{B}_1$, both A and B have only compact components (Neugebauer [7]). Clearly, the set

$$K = I - \bigcup \{Q^0: Q \text{ is a component of } A \text{ or } B\}$$

is perfect, and $K \supseteq C \cup (\partial A \cap \partial B)$, where ∂ is the boundary operator.

Moreover, both $A \cap K$ and $B \cap K$ are dense in K. To show this, let U be any open interval that meets K. Then $A \cup C$ and $B \cup C$ are F_σ-sets in the complete metric space $\bar{U} \cap K$, and hence, by the Baire Category Theorem, one of these sets, say $A \cup C$, contains a set $V \cap K$, where V is an open interval. Choose $a_1 \in A \cap K \cap V$. Since C is countable and K is uncountable, V must contain a point b_1 of B with, say, $b_1 < a_1$. If $\{b_1\}$ is a component of B, then clearly $b_1 \in \partial A \cap \partial B \subseteq K$, so that $B \cap K \neq \Lambda$. On the other hand, if $\{b_1\}$ is not a component of B, let $Q = [d, e]$ be the component of B that contains b_1. It follows then that $e < a_1$ and $e \in \partial A \cap \partial B \subseteq K$, so that $B \cap K \neq \Lambda$. Hence, $A \cap K$ and $B \cap K$ are dense in K.

The sets of discontinuities of $f \mid K$ and $g \mid K$ are of first category in K. Hence, the set of common points of continuity is of second category, hence uncountable. Therefore, we can choose an $x_0 \in K - C$ such that $f \mid K$ and $g \mid K$ are continuous at x_0. Now let $a_n \to x_0$ and $b_n \to x_0$, where $a_n \in A \cap K$ and $b_n \in B \cap K$. Then $g(x_0) = \lim g(a_n) \geq \lambda \geq \lim f(b_n) = f(x_0)$; this contradicts the fact that $g < f$ on I. Therefore $|C| = \mathfrak{c}$, and the lemma is proved.

We say that the subset A of $B \subseteq I$ is $\mathfrak{c}$*-dense* in B if each open set U meeting B meets A in $\mathfrak{c}$ points.

LEMMA 4. *Let* $\{A_m\}_{m=1}^{\infty}$ *be a sequence of sets that are ambiguous of class* 2 (*respectively, of additive class* $\alpha \geq 2$), *each* $\mathfrak{c}$*-dense in itself. Then there exists a sequence* $\{C_m\}_{m=1}^{\infty}$ *of disjoint sets, ambiguous of class* 2 (*respectively, ambiguous of class* $\alpha + 1$), *such that each* C_m *is* $\mathfrak{c}$*-dense in* A_m *and*

$$\bigcup_{m=1}^{\infty} A_m = \bigcup_{m=1}^{\infty} C_m.$$

Proof. From a theorem of Ceder [4] it follows that each $\mathfrak{c}$-dense in itself Borel set contains $\mathfrak{c}$ disjoint F_σ-sets, each $\mathfrak{c}$-dense in the given set. Let $\{A_{1\alpha}\}_{\alpha<\mathfrak{c}}$ be such a collection of F_σ-subsets of A_1. Fix $n > 1$. Then there are at most countably many ordinals α for which $A_n - A_{1\alpha}$ is not $\mathfrak{c}$-dense in A_2. For otherwise there would exist an uncountable set Γ such that $\alpha \in \Gamma$ implies $A_n - A_{1\alpha}$ is not $\mathfrak{c}$-dense in A_2. Hence, for each $\alpha \in \Gamma$ there would exist a rational interval U_α that meets A_2 and for which $|(A_n - A_{1\alpha}) \cap U| < \mathfrak{c}$. Thus there would exist some rational interval U and $\alpha, \beta \in \Gamma$ for which $\alpha \neq \beta$ and such that

$$|(A_n - A_{1\alpha}) \cap U| < \mathfrak{c} \quad \text{and} \quad |(A_n - A_{1\beta}) \cap U| < \mathfrak{c}.$$

This would contradict the fact that $A_{1\alpha}$ and $A_{1\beta}$ are disjoint and $\mathfrak{c}$-dense in A_1. Therefore, for each $n > 1$, there exists an $\alpha_n < \mathfrak{c}$ such that $A_n - A_{1\alpha}$ is $\mathfrak{c}$-dense in A_n whenever $\alpha \geq \alpha_n$.

Next, let γ be the least upper bound of the sequence $\{\alpha_n\}_{n=2}^{\infty}$. Then

$$A_{1\gamma},\ A_2 - A_{1\gamma},\ \cdots,\ A_n - A_{1\gamma},\ \cdots$$

are $\mathfrak{c}$-dense in $A_1, A_2, \cdots, A_n, \cdots$, respectively. Now we repeat the above argument on the sequence $\{A_n - A_{1\gamma}\}_{n=3}^{\infty}$, after finding in $A_2 - A_{1\gamma}$ $\mathfrak{c}$ disjoint F_σ-sets, each $\mathfrak{c}$-dense in $A_2 - A_{1\gamma}$.

Proceeding in this manner by induction, we obtain a sequence $\{B_n\}_{n=1}^{\infty}$ of disjoint F_σ-sets, each $\mathfrak{c}$-dense in A_n. We define

$$C_n = B_n \cup \left(A_n - \bigcup_{k=1}^{\infty} B_k - \bigcup_{i=1}^{n-1} A_i\right).$$

Then $C_n \cap C_m = \Lambda$ for $n \neq m$, $\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} A_n$, and each C_n is $\mathfrak{c}$-dense in A_n. Moreover, it is easily seen that if each A_n is ambiguous of class 2 (respectively, additive of class α), then each C_n will be ambiguous of class 2 (respectively, ambiguous of class $\alpha + 1$). This completes the proof.

LEMMA 5. *Let* A *be an ambiguous set of class* 2, $\mathfrak{c}$*-dense in itself, and let* J *be any closed interval. Then there exists a function* h *from* A *onto* J *such that* $h(B \cap A) = J$ *for each interval* B *that meets* A. *Moreover,* $h \in \mathscr{B}_2$, *and* $h^{-1}(G)$ *is ambiguous of class* 2 *for each open* G.

Proof. Let $\{B_n\}_{n=1}^{\infty}$ be a countable base of the family of intervals that meet A. Then we can easily find a sequence $\{C_n\}_{n=1}^{\infty}$ of disjoint sets, each homeomorphic to the Cantor set, such that $C_n \subseteq B_n$ for each n. Now choose h_n to be a continuous function (that is, a "Cantor function") from C_n onto J. Define $h = h_n$ on C_n, and for $x \notin \bigcup_{n=1}^{\infty} C_n$, let $h(x)$ be the midpoint of J. Then obviously $h(B \cap A) = J$, for each interval B that meets A. Moreover, for each open set G, we have the relation

$$h^{-1}(G) = \left(\bigcup_{n=1}^{\infty} h_n^{-1}(G)\right) \cup \left(A - \bigcup_{n=1}^{\infty} C_n\right);$$

the right-hand member is both an $F_{\sigma\delta}$- and a $G_{\delta\sigma}$-set, and the proof is complete.

THEOREM 3. *Let* $g < f$ *and* $f, g \in \mathscr{D}\mathscr{B}_1$. *Then there exists an* $h \in \mathscr{D}\mathscr{B}_2$ *such that* $g < h < f$ *and* $h^{-1}(G)$ *is ambiguous of class* 2, *for each open* G.

Proof. For rational numbers r and s with $r < s$, put

$$D_{r,s} = \{x: g(x) < r < s < f(x)\}.$$

Let $\Gamma = \{\langle a, b\rangle: |D_{a,b}| = \mathfrak{c}\}$. For each $\langle r, s\rangle \in \Gamma$, we can decompose $D_{r,s}$ into disjoint sets $E_{r,s}$ and $A_{r,s}$, the first countable, the second $\mathfrak{c}$-dense in itself. Since $D_{r,s}$ is an F_σ-set, $A_{r,s}$ is ambiguous of class 2. Define the set K by $K = I - \bigcup_\Gamma A_{r,s}$. Then K is countable and contains $\bigcup_\Gamma E_{r,s}$.

By Lemma 4, we can find a collection $\{C_{r,s}\}_{\langle r,s\rangle \in \Gamma}$ of disjoint sets, each ambiguous of class 2, such that each $C_{r,s}$ is a $\mathfrak{c}$-dense subset of $A_{r,s}$ and $\bigcup_\Gamma C_{r,s} = \bigcup_\Gamma A_{r,s} = I - K$. According to Lemma 5, we can find a function h_{rs} from $C_{r,s}$ onto $[r, s]$ such that, for each open interval B that meets $C_{r,s}$, $h_{rs}(B \cap C_{r,s}) = [r, s]$, and such that $h_{rs}^{-1}(G)$ is ambiguous of class 2 for each open G. Now define $h = \bigcup_\Gamma h_{rs}$.

Then h is a Darboux function over I - K (that is, it sends relative intervals into intervals). To show this, suppose $h(a) < \lambda < h(b)$. We want to find an $x \in [a, b] - K$ such that $h(x) = \lambda$. By Lemma 3, $\{z \in [a, b]: g(z) < \lambda < f(z)\}$ is an uncountable F_σ-set, and therefore, by Lemma 2, it contains a perfect set P on which both f and g are continuous. Pick $x \in P \cap (a, b)$, and select rationals r and s such that $g(x) < r < \lambda < s < f(x)$. Then obviously, for each neighborhood U of x, $|U \cap D_{r,s}| = \mathfrak{c}$ and hence $|U \cap C_{r,s}| = \mathfrak{c}$. So, by the construction of h_{rs}, there exists a $y \in U \cap C_{r,s}$ such that $h_{rs}(y) = h(y) = \lambda$.

Now we must define h on K so that h is a Darboux function over all of I. For $x \in K$, let h(x) be any point in the set

$$\{y: y = \lim_{n\to\infty} h(x_n) \text{ for } x_n \to x \text{ and } x_n \notin K \cup \{x\}\}.$$

Since $h \in \mathscr{U}$ on I - K, the latter set is easily seen to be a nonempty closed interval. With this extension of h to K, h clearly becomes a Darboux function on I.

Since obviously $g < h < f$, it remains only to prove that $h^{-1}(G)$ is both an $F_{\sigma\delta}$- and a $G_{\delta\sigma}$-set, for each open G. But

$$h^{-1}(G) = \bigcup_\Gamma h_{rs}^{-1}(G) \cup K_1,$$

where $K_1 \subseteq K$. Therefore, since each $h_{rs}^{-1}(G)$ is ambiguous of class 2, $h^{-1}(G)$ is also ambiguous of class 2. The proof of the theorem is complete.

Note that the h in Theorem 3 is a special $\mathscr{B}_2$-function, inasmuch as $h^{-1}(G)$ is not only a $G_{\delta\sigma}$- but also an $F_{\sigma\delta}$-set. It remains unknown whether Theorem 3 can be improved so that h may be taken to be a $\mathscr{D}\mathscr{B}_1$-function.

Theorem 3 cannot be extended so that it gives a $\mathscr{D}\mathscr{B}_{\alpha+1}$-function between two comparable $\mathscr{D}\mathscr{B}_\alpha$-functions. In fact, the following example shows that there need not exist $\mathscr{D}$-functions between two comparable $\mathscr{D}\mathscr{B}_2$-functions.

EXAMPLE 1. *There exist two comparable $\mathscr{D}\mathscr{B}_2$-functions admitting no $\mathscr{D}$-function between them.*

Proof. First we invoke the following lemma from [3]: Lemma B. *Let* A *be a Borel set* $\mathfrak{c}$*-dense in itself. Then* A *contains a sequence* $\{A_i\}_{i=1}^{\infty}$ *of disjoint* F_σ*-sets, each* $\mathfrak{c}$*-dense in* A; *therefore,* A *can be decomposed into the* $\mathfrak{c}$*-dense Borel sets* $\{A_i\}_{i=2}^{\infty}$ *and* $A_1 \cup \left(A - \bigcup_{i=2}^{\infty} A_i\right)$.

Applying this lemma to I, we can decompose I into $\bigcup_{i=1}^{4} A_i$, where each A_i is ambiguous of class 2 and $\mathfrak{c}$-dense in I. By a modified version of Lemma 5 in which J is an arbitrary interval, we can find functions k_1 and k_2 from A_1 and A_2, respectively, onto $(0, +\infty)$ and $(-\infty, 0)$, respectively, such that k_1 and k_2 take each positive (respectively, negative) value on each subinterval of I, and such that moreover, $k_i^{-1}(G)$ is ambiguous of class 2 for $i = 1, 2$, for each open set G.

Now define f and g as follows:

$$f(x) = \begin{cases} k_1(x) & \text{for } x \in A_1, \\ 2g(x) & \text{for } x \in A_2, \\ 0 & \text{for } x \in A_3, \\ 1 & \text{for } x \in A_4, \end{cases} \qquad g(x) = \begin{cases} k_2(x) & \text{for } x \in A_1, \\ \frac{1}{2} f(x) & \text{for } x \in A_2, \\ -1 & \text{for } x \in A_3, \\ 0 & \text{for } x \in A_4. \end{cases}$$

Then clearly $f, g \in \mathscr{D}\mathscr{B}_2$ and $g < f$. But 0 cannot be in the range of any function between g and f.

Although we cannot necessarily insert a $\mathscr{D}$-function between two comparable $\mathscr{D}\mathscr{B}_2$-functions, we can insert not only a $\mathscr{U}$-function, as guaranteed by Theorem 2, but also a $\mathscr{U}\mathscr{B}_3$-function, as the following theorem demonstrates. The theorem is stated so as to exclude the case where $\alpha = 1$; for in this case the assertion would be considerably weaker than that of Theorem 3, because, although $\mathscr{U}\mathscr{B}_1 = \mathscr{D}\mathscr{B}_1$, the class $\mathscr{D}\mathscr{B}_2$ is a proper subset of $\mathscr{U}\mathscr{B}_2$ (see [2] or [3]).

THEOREM 4. *Let* $g < f$ *and* $f, g \in \mathscr{U}\mathscr{B}_\alpha$, *where* $\alpha \geq 2$. *Then there exists an* $h \in \mathscr{U}\mathscr{B}_{\alpha+1}$ *such that* $g < h < f$ *and such that* $h^{-1}(G)$ *is ambiguous of class* $\alpha + 1$ *for each open* G.

Proof. The proof is similar to that of Theorem 3. As in that proof, form the sets $D_{r,s}$, $A_{r,s}$, and $C_{r,s}$. Each $D_{r,s}$, and hence each $A_{r,s}$, is of additive class α, and by Lemma 4, each $C_{r,s}$ is of ambiguous class $\alpha + 1$. We shall select a function h_{rs} mapping $C_{r,s}$ onto the rationals in $[r, s]$ so that $h_{rs}(B \cap C_{r,s}) = h_{rs}(C_{r,s})$ for each open B that meets $C_{r,s}$, so that $h_{rs}^{-1}(G)$ is ambiguous of class $\alpha + 1$ for each open G, and so that $h^{-1}(\lambda)$ is empty or has cardinality $\mathfrak{c}$. This can be done as follows: by Lemma B, we can decompose each $C_{r,s}$ into countably many disjoint sets $\{C_n\}_{n=1}^{\infty}$, where each C_n is ambiguous of class $\alpha + 1$ and $\mathfrak{c}$-dense in $C_{r,s}$. Define h_{rs} to map C_n onto the nth rational in $[r, s]$. Then h_{rs} has the desired properties.

As before, define $h = \bigcup_{\Gamma} h_{rs}$ on I - K. To show that $h \in \mathscr{U}$ relative to I - K, let J be an open interval inside $(h(a), h(b))$, and let $|C| < \mathfrak{c}$. By Lemma 1, there exists a rational $\lambda \in J$ such that $|E| = \mathfrak{c}$, where $E = \{z \in [a, b]: g(z) < \lambda < f(z)\}$. By Lemma 2, there exists a perfect subset P of E such that $f \mid P$ and $g \mid P$ are continuous. Now pick $x \in P \cap (a, b)$ and rationals r, s so that

$$g(x) < r < \lambda < s < f(x).$$

As before, $|C_{r,s} \cap U| = \mathfrak{c}$ for each neighborhood of x. Hence, $|h_{rs}^{-1}(\lambda) \cap U| = \mathfrak{c}$. Therefore $h^{-1}(J) \cap ([a, b] - C) \neq \Lambda$; this shows that $h \in \mathscr{U}$ over I - K.

Again, we extend h to I, and it is easily verified that $h \in \mathscr{U}\mathscr{B}_{\alpha+1}$, that $g < h < f$, and that $h^{-1}(G)$ is ambiguous of class $\alpha + 1$, for each open G.

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