GENERALIZED GREEN'S MATRICES FOR COMPATIBLE DIFFERENTIAL SYSTEMS

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1. INTRODUCTION

The existence and properties of a Green's matrix for an incompatible differential system with complex-valued coefficients are well known. Generalized Green's functions for compatible differential systems consisting of a single nth-order linear differential equation with continuous coefficients together with boundary conditions involving the values of the solution and its first $n - 1$ derivatives at two points may be traced back to Hilbert [7]. Elliot [5] made a systematic study of such systems, and Reid [8] extended Elliot's results to compatible first-order systems involving n differential equations and the same number of boundary conditions. Recently, the generalized Green's function for the case involving the single nth-order linear differential equation and boundary conditions has been treated by Greub and Rheinboldt [6] and by Wyler [11]; a more comprehensive treatment of an algebraic theory of operator solutions of boundary problems, which includes this case as a special instance, is given by Wyler [12]. More recently, Bradley [3] has extended the results of Elliot to quasi-differential operators of Euler type where the number of boundary conditions is not necessarily the same as the order of the operator. The purpose of this paper is to extend these latter results to vector-matrix systems, and thus to generalize the theorems of Reid [8].

The specific form of the operator and some basic relationships that exist between an operator and its adjoint are given in Section 2. Existence of a generalized Green's matrix is established in Section 3, and Section 4 is concerned with the existence and properties of a "principal" generalized Green's matrix. In particular, it is shown that with respect to certain orthogonality conditions there exists a unique generalized Green's matrix that is characterized by conditions similar to those characterizing the unique Green's matrix for an incompatible system. Finally, Section 5 is devoted to a brief discussion of boundary-value problems equivalent under a nonsingular transformation, and to an extension of a result of Reid [9] which for incompatible systems relates the Green's matrix for one system to the Green's matrix of an equivalent system by a functional equation involving the respective leading coefficients and the nonsingular transformation matrix.

Matrix notation is used throughout; in particular, matrices with one column are called vectors. The $p \times p$ identity matrix is denoted by E_p, and 0 is used indiscriminately for the zero matrix of any dimension; the conjugate transpose of a matrix M is denoted by M^*. If the elements of a matrix function M are absolutely continuous on an interval $[a, b]$, then $M'(x)$ denotes the matrix of derivatives at those values of x for which the derivatives exist, and zero elsewhere; similarly, if M is integrable (Lebesgue), then $\int_a^b M$ denotes the matrix of integrals. A matrix function is called continuous, differentiable, and so forth, whenever each element has that property. Finally, if u and v are vector functions with the property that $v^* u$ is integrable, then the integral inner product $\int_a^b v^* u$ is denoted by (u, v).

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2. ADJOINT OPERATORS

Let A_1 and A_0 denote $n \times n$ matrix functions, A_1 being absolutely continuous and nonsingular, and A_0 being integrable on an interval $[a, b]$. Then L will denote the linear differential operator whose domain is the collection $\mathscr{A}$ of all absolutely continuous n-dimensional vector functions y and which is defined by the formula

$$L[y] = A_1 y' + A_0 y. \tag{2.1}$$

For the operator L^+ that is defined by

$$L^+[z] = (-A_1^* z)' + A_0^* z, \tag{2.2}$$

we obtain immediately the Lagrange identity:

LEMMA 2.1. *If* $y, z \in \mathscr{A}$, *then* $z^* L[y] - (L^+[z])^* y = (z^* A_1 y)'$.

Define $\mathscr{D}_0$ to be the collection of vector functions y in $\mathscr{A}$ for which $y(a) = 0 = y(b)$, and denote by $\mathscr{D}_0^*$ the collection of n-dimensional vector functions z that are essentially bounded and measurable and for which there exists an integrable n-dimensional vector function f_z such that $(L[y], z) - (y, f_z) = 0$ for all $y \in \mathscr{D}_0$. The following theorem then is a consequence of Lemma 2.1 and the fundamental lemma of the calculus of variations.

THEOREM 2.1. $\mathscr{D}_0^* = \mathscr{A}$ *and* $z \in \mathscr{D}_0^*$ *implies that* $f_z = L^+[z]$.

If T_0 is the restriction of L to $\mathscr{D}_0$, then the adjoint operator for T_0 is the operator T_0^* with domain $\mathscr{D}_0^*$ that is defined by $T_0^*[z] = f_z$. It follows from Theorem 2.1 that $T_0^* = L^+$.

Let $\mathscr{D}$ be a linear subspace of $\mathscr{A}$ that contains $\mathscr{D}_0$, and let T be the restriction of L to $\mathscr{D}$. Denote by $\mathscr{D}^*$ the collection of essentially bounded and measurable n-dimensional vector functions z for which there is an integrable n-dimensional vector function f_z such that $(T[y], z) - (y, f_z) = 0$ for all $y \in \mathscr{D}$. It follows that $\mathscr{D}^* \subset \mathscr{A}$, and for each z in $\mathscr{D}^*$ there is at most one f_z, namely $L^+[z]$, such that $(T[y], z) - (y, f_z) = 0$ for all y in $\mathscr{D}$. The adjoint T^* of T is the operator on $\mathscr{D}^*$ defined by the formula $T^*[z] = f_z$; that is, T^* is the restriction of L^+ to $\mathscr{D}^*$. The operator T is said to be self-adjoint if and only if $\mathscr{D} = \mathscr{D}^*$ and $T = T^*$.

Let $\mathscr{B}$ and $\mathscr{B}^*$ represent the respective subspaces of 2n-dimensional complex space of end values $(y(a); y(b))$ for vector functions y in $\mathscr{D}$ and $\mathscr{D}^*$, respectively. If $k < 2n$ and the dimension of $\mathscr{B}$ is $2n - k$, then P and Q will denote $n \times (2n - k)$ matrices with the property that the columns of $[-P^* Q^*]^*$ form a basis for $\mathscr{B}$. If, in addition, $k > 0$, let M and N be $k \times n$ matrices such that the $k \times 2n$ matrix $[MN]$ has rank k and $MP - NQ = 0$. Then $\mathscr{D}$ is characterized as the collection of vector functions y in $\mathscr{A}$ with the property that $My(a) + Ny(b) = 0$. If $k = 0$, then $\mathscr{D} = \mathscr{A}$. Moreover, $\dim \mathscr{B} + \dim \mathscr{B}^* = 2n$, and if $\dim \mathscr{B} = m > 0$, then $\mathscr{D}^*$ is the collection of vector functions z in $\mathscr{A}$ for which

$$P^* A_1^*(a) z(a) + Q^* A_1^*(b) z(b) = 0.$$

More generally, if $\dim \mathscr{B} = m > 0$ and R and S are $(2n - m) \times n$ matrices, then $\mathscr{D}^*$ is the collection of vector functions z in $\mathscr{A}$ for which $Rz(a) + Sz(b) = 0$ if and only if the $(2n - m) \times 2n$ matrix $[R\ S]$ has rank $2n - m$ and

$$MA_1^{*-1}(a) R^* - NA_1^{*-1}(b) S^* = 0.$$

It follows from the foregoing remarks that the adjoint of T^* is T.

The nullity of the operator T is the dimension of the null space of T, that is, the dimension of $\{y: y \in \mathscr{D},\ L[y] = 0\}$. The next two theorems follow from classical results [4, Chapter 11].

THEOREM 2.2. *If* $\dim \mathscr{B}^* = k$ *and the nullity of* T *is* r, *then* $k + r - n$ *is the nullity of* T^*.

THEOREM 2.3. *An integrable* n*-dimensional vector function* f *is in the range of* T *if and only if* $(f, z) = 0$ *for all* z *in the null space of* T^*.

It should be noted that if $A_1(x) = \mathscr{J}$ for all $x \in [a, b]$, where

$$\mathscr{J} = \begin{bmatrix} 0 & -E_\mu \\ E_\nu & 0 \end{bmatrix},$$

then the operators (2.1) and (2.2) have precisely the form (2.8) and (2.9), respectively, of [3]; these in turn are equivalent to a quasi-differential operator and its adjoint. Thus the theorems in Sections 3 and 4 of this paper extend the results of Section 5 of [3]. If, moreover, $\mu = \nu$, then (2.1) and (2.2) are of the form studied by Reid in [10].

3. EXISTENCE OF A GENERALIZED GREEN'S MATRIX

By definition, a generalized Green's matrix for the operator T is an essentially bounded and measurable $n \times n$ matrix function G on

$$\square \equiv \{(x, t): a \le x \le b;\ a \le t \le b\}$$

with the property that if f is in the range of T, then the function y defined by

$$y(x) = \int_a^b G(x, t) f(t)\, dt$$

is in $\mathscr{D}$ and $T[y] = f$. We shall now construct a matrix function by following the classical method for constructing a Green's matrix for the case where T and T^* have nullity zero, but using E. H. Moore's generalized inverse of a matrix.

LEMMA 3.1. *If* Y *is a fundamental matrix for the equation* $L[y] = 0$, *then there exists a fundamental matrix* Z *for* $L^+[z] = 0$ *such that* $Z^*(x) A_1(x) Y(x) = E_n$ *for all* $x \in [a, b]$.

This result follows from Lemma 2.1 and the existence and uniqueness theorems for first-order vector-matrix differential equations.

THEOREM 3.1. *If* f *is an integrable* n*-dimensional vector function,* $s \in [a, b]$, c *is a constant* n*-vector, and* Y *and* Z *are as in Lemma* 3.1, *then the solution* y *of* $L[y] = f$ *satisfying* $y(s) = Y(s)c$ *is given by*

$$y(x) = Y(x)c + \int_s^x Y(x) Z^*(t) f(t)\, dt.$$

Indeed, if we set $y(x) = Y(x)u(x)$, then y is a solution of the system $L[y] = f$, $y(s) = Y(s)c$ if and only if $A_1 Yu' = f$, $u(s) = c$. Hence $u' = Z^* f$ and

$$u(x) = c + \int_s^x Z^*(t) f(t)\, dt,$$

from which the theorem follows.

Now suppose that Y is a fundamental matrix for $L[y] = 0$ and that Z is chosen as in Lemma 3.1. For $k > 0$ and $\dim \mathscr{B} = 2n - k$, let $s(Y)$ and $s^-(Y)$ be $k \times n$ matrices defined as $s(Y) = MY(a) + NY(b)$ and $s^-(Y) = MY(a) - NY(b)$. If r is the nullity of T, then $s(Y)$ has rank $n - r$. If $r > 0$, let B be an $n \times r$ matrix with the property that $B^* B = E_r$ and $s(Y)B = 0$. If $r > n - k$, then C will represent a $k \times (k - n + r)$ matrix such that $C^* C = E_{k-n+r}$ and $C^* s(Y) = 0$. It follows that the $(k + r) \times (k + r)$ matrix

$$\begin{bmatrix} s(Y) & C \\ B^* & 0 \end{bmatrix} \tag{3.1}$$

is nonsingular and its inverse is of the form

$$\begin{bmatrix} R & B \\ C^* & 0 \end{bmatrix}. \tag{3.2}$$

The $n \times k$ matrix R is the generalized inverse of $s(Y)$ in the sense of E. H. Moore (for a more complete discussion, see [10, Section 6]). If $r = 0$, then the matrix B does not appear; if $r = n - k$, then C does not appear.

Now if $\dim \mathscr{B} < 2n$, let $G_0(x, t)$ be the $n \times n$ matrix defined by

$$G_0(x, t) = \frac{1}{2} Y(x)\left[\frac{|x - t|}{x - t} E_n + Rs^-(Y)\right] Z^*(t) \qquad \text{when } x \neq t \text{ and } x, t \in [a, b],$$

$$G_0(x, x) = \frac{1}{2} Y(x) Rs^-(Y) Z^*(x) \quad \text{when } x \in [a, b].$$

If $\dim \mathscr{B} = 2n$, let $G_0(x, t)$ be defined by

$$G_0(x, t) = \frac{1}{2} \frac{|x - t|}{x - t} Y(x) Z^*(t) \qquad \text{when } x \neq t \text{ and } x, t \in [a, b],$$

$$G_0(x, x) = 0 \qquad \text{when } x \in [a, b].$$

If r is the nullity of T and $r = n - k$, then it follows from Theorem 2.2 that the nullity of T^* is 0. Hence, by virtue of Theorem 2.3, the range of T is precisely the collection of all integrable n-dimensional vector functions. For $r > n - k$, we have the following theorem.

THEOREM 3.2. *Let* Y *and* Z *be as in Lemma* 3.1, *let the nullity* r *of* T *satisfy* $r > n - k$, *and suppose that* C *is a* $k \times (k - n + r)$ *matrix satisfying*

$$C^* C = E_{k-n+r} \quad \textit{and} \quad C^* s(Y) = 0.$$

Then an integrable n-*dimensional vector function* f *is in the range of the operator* T *if and only if*

$$C^* s^-(Y) \int_a^b Z^*(t) f(t)\, dt = 0. \tag{3.3}$$

Now it follows from Theorem 3.1 that if y is a solution of $L[y] = f$, then for a suitable vector ξ we have the relation

$$y(x) = \frac{1}{2}\left[Y(x)\xi + \int_a^x Y(x) Z^*(t) f(t)\, dt - \int_x^b Y(x) Z^*(t) f(t)\, dt \right]. \tag{3.4}$$

If, moreover, $y \in \mathscr{D}$, then $My(a) + Ny(b) = 0$,

$$s(Y)\xi = s^-(Y) \int_a^b Z^*(t) f(t)\, dt, \tag{3.5}$$

and therefore (3.3) follows. On the other hand, if (3.3) holds, define ξ as

$$\xi = R s^-(Y) \int_a^b Z^*(t) f(t)\, dt,$$

and it is evident that $S^* \xi = 0$ and (3.5) holds. Now, if y is defined by the formula (3.4), then $L[y] = f$ and $y \in \mathscr{D}$, since $My(a) + Ny(b) = 0$.

THEOREM 3.3. *The matrix function* G_0 *defined above is a generalized Green's function for* T.

If $\dim \mathscr{B} = 2n$, then this result follows directly from Theorem 3.1.

If $\dim \mathscr{B} < 2n$, then Theorem 3.3 follows from Theorem 3.2; in particular, set

$$\xi = R s^-(Y) \int_a^b Z^*(t) f(t)\, dt$$

in (3.4), where R is the generalized inverse of S(Y).

In the sequel, the number ρ will be defined by the equation $\rho = k + r - n$.

THEOREM 3.4. *The generalized Green's matrix for* T *is not unique. If the columns of the* $n \times r$ *matrix function* U *form a basis for the null space of* T *and the columns of the* $n \times \rho$ *matrix function* V *form a basis for the null space of* T^*, *and* G_1 *is one generalized Green's matrix for* T, *then a matrix function* G *on* $\square$ *is also a generalized Green's matrix for* T *if and only if there exist essentially bounded and measurable* $r \times n$ *and* $n \times \rho$ *matrix functions* Γ *and* Λ, *respectively, such that*

$$G(x, t) = G_1(x, t) + U(x)\Gamma(t) + \Lambda(x) V^*(t). \tag{3.6}$$

If Γ and Λ are essentially bounded and measurable $r \times n$ and $n \times \rho$ matrix functions, respectively, and G is defined by (3.6), then it is clear from Theorem 2.3 that G is a generalized Green's matrix for T.

To establish the converse, we may assume without loss of generality that

$$\int_a^b U^* U = E_r \quad \text{and} \quad \int_a^b V^* V = E_\rho .$$

If w is an integrable n-dimensional vector function and

$$f(x) = w(x) - V(x) \int_a^b V^*(t) w(t) \, dt ,$$

then $\int_a^b V^*(x) f(x) \, dx = 0$. Thus it follows Theorem 2.3 that f is in the range of T. Suppose that G is a second generalized Green's matrix for T, and let

$$F(x, t) = G(x, t) - G_1(x, t) .$$

Then there exists a constant r-vector ξ such that

$$\int_a^b F(x, t) f(t) \, dt = U(x) \xi ,$$

and if

$$K(x, t) = F(x, t) - \int_a^b F(x, s) V(s) V^*(t) \, ds ,$$

then $\int_a^b \int_a^b U^*(x) K(x, t) f(t) \, dt \, dx = \xi$, and consequently

$$\int_a^b \left[K(x, t) - U(x) \int_a^b U^*(s) K(s, t) \, ds \right] w(t) \, dt = 0 .$$

But w is an arbitrary integrable vector function; therefore

$$K(x, t) - U(x) \int_a^b U^*(s) K(s, t) \, ds = 0 \quad \text{on } \square$$

and

$$F(x, t) = U(x) \int_a^b U^*(s) K(s, t) \, ds + \int_a^b F(x, s) V(s) V^*(t) \, ds .$$

Hence, (3.6) holds, with Γ and Λ defined by the equations

$$\Gamma(t) = \int_a^b U^*(s) K(s, t) \, ds \quad \text{and} \quad \Lambda(x) = \int_a^b F(x, s) V(s) \, ds .$$

It is clear that Γ and Λ are essentially bounded and measurable.

COROLLARY I. *If* f *is in the range of* T *and* y *is a function in* $\mathcal{D}$ *such that* $T[y] = f$, *then there exists a generalized Green's matrix* G *for* T *such that*

$$y(x) = \int_a^b G(x, t) f(t)\, dt \qquad \text{on } [a, b].$$

The function y does not, however, determine G uniquely, since

$$G_1(x, t) = G(x, t) + \Lambda(x) V^*(t)$$

may be different from G(x, t); but

$$y(x) = \int_a^b G_1(x, t) f(t)\, dt = \int_a^b G(x, t) f(t)\, dt.$$

In fact, we have the following result, modulo the definition of a generalized Green's function along the diagonal.

COROLLARY II. *If* f *is an integrable vector function, then there exists a one-to-one correspondence between the functions* y *in* $\mathcal{D}$ *for which* $T[y] = f$ *and the generalized Green's functions for* T *if and only if the nullity of* T^* *is zero.*

4. PRINCIPAL GENERALIZED GREEN'S MATRICES

We shall now give conditions that determine a unique generalized Green's matrix.

THEOREM 4.1. *Let* U *and* V *satisfy the hypothesis in Theorem* 3.4, *and suppose that* Θ *and* Ω *are integrable* $n \times r$ *and* $n \times \rho$ *matrix functions, respectively, with the property that the matrices*

$$\int_a^b \Theta^* U \qquad \textit{and} \qquad \int_a^b V^* \Omega$$

are nonsingular. Then there exists a unique generalized Green's matrix $G_{\Theta\Omega}$ *for* T *satisfying the conditions*

$$\begin{aligned} &\int_a^b G_{\Theta\Omega}(x, t)\Omega(t)\, dt = 0 \qquad (x \in [a, b]), \\ (4.1) \qquad &\int_a^b \Theta^*(x) G_{\Theta\Omega}(x, t)\, dx = 0 \qquad (t \in [c, b]). \end{aligned}$$

We may assume that $\int_a^b \Theta^* U = E_r$ and $\int_a^b V^* \Omega = E_\rho$. Suppose G_0 is the generalized Green's matrix described in Theorem 3.3 and determines matrix functions Γ and Λ such that the generalized Green's matrix given by (3.6) satisfies conditions

(4.1). Such a generalized Green's matrix G satisfies the conditions (4.1) if and only if Γ and Λ satisfy the equations

$$\text{(4.2)}\qquad \begin{aligned} &\Gamma(x) + \int_a^b \Theta^*(s)G_0(s, x)\,ds + \int_a^b \Theta(s)\Lambda(s)V^*(x)\,ds = 0, \\ &\Lambda(x) + \int_a^b G_0(x, s)\Omega(s)\,ds + \int_a^b U(x)\Gamma(s)\Omega(s)\,ds = 0. \end{aligned}$$

A particular pair of solutions for equations (4.2) is

$$\Lambda(x) = -\int_a^b G_0(x, s)\Omega(s)\,ds,$$

$$\Gamma(x) = \int_a^b \int_a^b \Theta(s)G_0(s, t)\Omega(t)V^*(x)\,ds\,dt - \int_a^b \Theta^*(s)G_0(s, x)\,ds.$$

Hence, the existence of $G_{\Theta\Omega}$ follows. Moreover, if Γ and Λ constitute a solution of (4.2), then after substituting the value of $\Gamma(x)$ given by the first equation into the second equation of (4.2), we can show by straightforward computation that the value of

$$U(x)\Gamma(t) + \Lambda(x)V^*(t)$$

is independent of the particular choice of Γ and Λ. Hence $G_{\Theta\Omega}$ is unique.

Following the terminology of Reid [8], we call a generalized Green's matrix satisfying (4.1) a *principal generalized Green's matrix for* T.

The following is a dual of Theorem 4.1.

THEOREM 4.2. *If* U, V, Θ, *and* Ω *satisfy the hypotheses in Theorem* 4.1, *then there exists a unique generalized Green's matrix* $H_{\Omega\Theta}$ *for* T^* *satisfying the conditions*

$$\text{(4.3)}\qquad \begin{aligned} &\int_a^b \Omega^*(x)H_{\Omega\Theta}(x, t)\,dx = 0 \qquad (t \in [a, b]), \\ &\int_a^b H_{\Omega\Theta}(x, t)\Theta(t)\,dt = 0 \qquad (x \in [a, b]). \end{aligned}$$

For brevity, denote by B_Ω and B_Θ the $n \times n$ matrix functions defined on $\square$ by

$$B_\Omega(x, t) = \Omega(x)V^*(t), \qquad B_\Theta(x, t) = \Theta(x)U^*(t),$$

and if $t \in [a, b]$, let $G_t(x) = G_{\Theta\Omega}(x, t)$.

THEOREM 4.3. *If* $G_{\Theta\Omega}$ *is the unique generalized Green's matrix satisfying* (4.1), *then the following five conditions are satisfied:*

(4.4) $G_{\Theta\Omega}$ *is continuous on* $\square$, *except along the line* $x = t$; *if* $t \in (a, b)$, *then* G_t *is differentiable on* $[a, t) \cup (t, b]$;

(4.5) *if* $t \in (a, b)$, *then* G_t *has right and left limits* $G_t(t^+)$ *and* $G_t(t^-)$, *respectively, at* t, *and*

$$G_t(t^+) - G_t(t^-) = A_1^{-1}(t);$$

(4.6) *if* $t \in [a, b]$, *then* $L[G_t](x) = -B_\Omega(x, t)$ *on* $[a, t) \cup (t, b]$;

(4.7) *if* $t \in (a, b)$, *then* G_t *satisfies the boundary conditions that characterize the set* $\mathscr{D}$;

$$\int_a^b \Theta^*(x) G_{\Theta\Omega}(x, t)\, dx = 0. \tag{4.8}$$

Conditions (4.4) to (4.6) may be verified directly, by means of the properties of G_0. Condition (4.8) is one of the conditions in (4.1). If $\mathscr{D} = \mathscr{A}$, then (4.7) is trivially satisfied. Otherwise, let w be an integrable vector function, and define f by

$$f(x) = w(x) - \Omega(x) \int_a^b V^*(t) w(t)\, dt.$$

It follows that $\int_a^b V^* f = 0$ and hence f is in the range of T. Let u be defined by

$$u(x) = \int_a^b G_{\Theta\Omega}(x, t) f(t)\, dt.$$

It follows from (4.1) that

$$\int_a^b G_{\Theta\Omega}(x, t) f(t)\, dt = \int_a^b G_{\Theta\Omega}(x, t) w(t)\, dt,$$

and therefore

$$0 = Mu(a) + Nu(b) = \int_a^b [MG_{\Theta\Omega}(a, t) + NG_{\Theta\Omega}(b, t)] w(t)\, dt,$$

from which (4.7) follows since w is an arbitrary integrable n-dimensional vector function.

COROLLARY. *If* w *is an integrable* n-*dimensional vector function and* y *is defined by* $y(x) = \int_a^b G_{\Theta\Omega}(x, t) w(t)\, dt$, *then*

$$L[y](x) = w(x) - \int_a^b B_\Omega(x, t) w(t)\, dt, \qquad y \in \mathscr{D}, \qquad \int_a^b \Theta^* y = 0. \tag{4.9}$$

It is clear that the function $G_{\Theta\Omega}$ satisfies (4.9). To see that these conditions follow from (4.4) to (4.8), let G be a function defined on $\square$ satisfying these latter conditions. The last two assertions of (4.9) follow from (4.7) and (4.8), respectively. The equation in (4.9) is equivalent to the assertion that G is a generalized Green's matrix for T. To show that this is the case, let

$$F(x, t) = G(x, t) - G_{\Theta\Omega}(x, t) \qquad \text{for } x \neq t,$$

$$F(t, t) = G(t^+, t) - G_{\Theta\Omega}(t^+, t),$$

and then let $F_t(x) = F(x, t)$. Condition (4.5) implies that F_t is continuous for each $t \in (a, b)$, and (4.6) implies that

$$A_1(x)\, F_t(x) = A_1(x)\, F_t(a) - \int_a^x A_0(s)\, F_t(x)\, ds,$$

from which it follows that there exists a continuous $r \times n$ matrix function Γ such that $F(x, t) = U(x)\,\Gamma(t)$. Thus it follows from Theorem 3.4 that G is a generalized Green's matrix for T.

Similarly, the unique generalized Green's matrix $H_{\Omega\Theta}$ for T^* that satisfies (4.3) also satisfies conditions analogous to (4.4) to (4.8).

THEOREM 4.4. *If* $x, t \in [a, b]$ *and* $x \neq t$, *then* $H_{\Omega\Theta}(x, t) = G^*_{\Theta\Omega}(t, x)$.

Let u and w be integrable n-dimensional vector functions, and define y and z by

$$y(x) = \int_a^b G_{\Theta\Omega}(x, t)\, u(t)\, dt, \qquad z(x) = \int_a^b H_{\Omega\Theta}(x, t)\, w(t)\, dt,$$

respectively. Then use the corollary to Theorem 4.3 and the definition of D^* to conclude that

$$\int_a^b \int_a^b w^*(t)\,[H^*_{\Omega\Theta}(x, t) - G_{\Theta\Omega}(t, x)]\, u(x)\, dx\, dt = 0.$$

The theorem follows, since u and w are arbitrary integrable vector functions.

COROLLARY I. *The matrix function* $G_{\Theta\Omega}$ *is characterized by the conditions* (4.4) *to* (4.8), *and* $H_{\Omega\Theta}$ *is characterized by analogous conditions.*

As a consequence of Theorems 3.4 and 4.4, we have the following result:

COROLLARY II. *If* G *is a generalized Green's matrix for* T, *then the function* H *defined by* $H(x, t) = G^*(t, x)$ *is a generalized Green's matrix for* T^*.

5. EQUIVALENT BOUNDARY PROBLEMS

Consider a second boundary problem

$$L^\circ[u] \equiv A_1^\circ u' + A_0^\circ u = 0 \qquad \text{and} \qquad u \in \mathscr{D}^\circ,$$

where A_1^o and A_1^1 satisfy the conditions specified for A_1, A_0, and $\mathscr{D}^o$ is either all of $\mathscr{A}$ or is characterized by the boundary conditions $M^o u(a) + N^o u(b) = 0$, where M^o and N^o are matrices of order $k \times n$ such that $[M^o \ N^o]$ has rank k. Let T^o be the restriction of L^o to $\mathscr{D}^o$. Following the terminology of Reid [9] and Bliss [1], [2], we say that the operator T is equivalent to the operator T^o under the transformation

$$u = Hy \tag{5.1}$$

provided H is a nonsingular, absolutely continuous matrix function on [a, b] such that a vector function y satisfies $L[y] = 0$ or belongs to $\mathscr{D}$ if and only if the corresponding u given by (5.1) satisfies $L^o[u] = 0$ or belongs to $\mathscr{D}^o$. It is clear that if T is equivalent to T^o under the transformation (5.1), then the nullity of T is the same as the nullity of T^o.

A classical result in the theory of equivalent boundary problems is that if $A_1(x) = A(x) = E$ for all $x \in [a, b]$, T is equivalent to T^o under (5.1), the number k of boundary conditions characterizing $\mathscr{D}$ and $\mathscr{D}^o$ is n, and the nullity of T (and therefore of T^o) is zero, then the unique Green's matrices G and G^o for T and T^o, respectively, satisfy the relation

$$H(x)G(x, t) = G^o(x, t)H(t) \qquad \text{if } x, t \in [a, b] \text{ and } x \neq t.$$

Reid [9] has extended this to the following theorem.

THEOREM 5.1. *If* T *is equivalent to* T^o *under the transformation* (5.1), $k = n$, G *and* G^o *are the Green's matrices for* T *and* T^o, *respectively*, $x, t \in [a, b]$, *and* $x \neq t$, *then*

$$H(x)G(x, t)A_1(t) = G^o(x, t)A_1^o(t)H(t).$$

If the condition that $k = n$ is removed, then the generalized Green's matrix is not unique; however, we have the following proposition.

THEOREM 5.2. *If* T *is equivalent to* T^o *under the transformation* (5.1) *and* G *is a generalized Green's matrix for* T, *then there exists a generalized Green's matrix* G^o *for* T^o *satisfying*

$$H(x)G(x, t)A_1(t) = G^o(x, t)A_1^o(t)H(t) \qquad \text{if } x, t \in [a, b] \text{ and } x \neq t.$$

It is sufficient to show that if

$$G^o(x, t) = H(x)G(x, t)A_1(t)H^{-1}(t)A_1^{o-1}(t) \qquad \text{for } x \neq t,$$

then G^o is a generalized Green's matrix for T^o. But this follows immediately from the fact that the identity

$$L^o[Hy] = A_1^o HA_1^{-1} L[y]$$

is equivalent to the condition that $L[y] = 0$ if and only if $L^o[Hy] = 0$. (See Reid [9].)

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