ON THE DIAMETER OF A GRAPH

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A *graph* G_n consists of a set of n *nodes*, some pairs of which are joined by a single *edge*. The *degree* of a node x is the number d(x) of edges joining it to other nodes. A graph is *connected* if it cannot be represented as the union of disjoint smaller graphs. The *diameter* of a connected graph G_n is the least integer k such that any pair of nodes of G_n can be joined by a sequence of at most k edges, consecutive ones of which have a node in common.

In what follows, n and k denote integers satisfying the inequality $n - 1 > k > 2$. Let g(n, k) be the least integer r such that if $d(x) \geq r$ for every node x of a connected graph G_n, then the diameter of G_n is at most k. The object of this note is to prove the following result (here [x] denotes the largest integer not exceeding x).

THEOREM.

$$g(n, k) = \begin{cases} \left[\frac{n}{t}\right] & \text{if } k = 3t - 4, \\ \left[\frac{n-1}{t}\right] & \text{if } k = 3t - 3, \\ \left[\frac{n-2}{t}\right] & \text{if } k = 3t - 2. \end{cases}$$

Proof. Let us suppose that $k = 3t - 3$, where t is an integer greater than 1. We first show that

$$g(n, k) \leq \left[\frac{n-1}{t}\right].$$

To accomplish this we assume the contrary, namely, that there exists a connected graph G_n, the degree of each of whose nodes is at least $\left[\frac{n-1}{t}\right]$ and whose diameter exceeds k. From this we shall deduce a contradiction.

It is easy to see that by introducing additional edges, we can transform G_n into a graph G'_n of the form

$$\langle 1 \rangle — \langle a_1 \rangle — \langle a_2 \rangle — \cdots — \langle a_{3t-3} \rangle — \langle 1 \rangle$$

(here $\langle j \rangle$ denotes a (nonempty) complete subgraph with j nodes and $\binom{j}{2}$ edges, and two nodes in different indicated subgraphs are joined by an edge if and only if the two subgraphs are adjacent in our representation), in such a way that the diameter of G'_n is $k + 1$ and the degree of each node of G'_n is at least $\left[\frac{n-1}{t}\right]$.

Consider such a graph. Clearly, the degree of the first node on the left is a_1. The degree of any node in $\langle a_3 \rangle$ is $a_2 + (a_3 - 1) + a_4$. Continuing in this fashion,

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we obtain the following set of inequalities, since by hypothesis the degree of every node is at least $\left[\frac{n-1}{t}\right]$.

$$\begin{aligned} a_1 &\geq \left[\frac{n-1}{t}\right], \\ a_2 + (a_3 - 1) + a_4 &\geq \left[\frac{n-1}{t}\right], \\ a_5 + (a_6 - 1) + a_7 &\geq \left[\frac{n-1}{t}\right], \\ &\cdots \\ a_{3t-7} + (a_{3t-6} - 1) + a_{3t-5} &\geq \left[\frac{n-1}{t}\right], \\ a_{3t-3} &\geq \left[\frac{n-1}{t}\right]. \end{aligned}$$

Upon adding and simplifying, we conclude that

$$n - t - a_{3t-4} \geq t\left[\frac{n-1}{t}\right] > n - 1 - t.$$

But this is false, since $a_{3t-4} \geq 1$; hence, the above upper bound for $g(n, k)$ is established.

If we can exhibit a connected graph G_n the degree of each of whose nodes is at least $\left[\frac{n-1}{t}\right] - 1$ and whose diameter is $k + 1$, then it will follow that

$$g(n, k) \geq \left[\frac{n-t}{t}\right].$$

To do this, consider a graph of diameter $k + 1$ whose structure is as illustrated above. Let

$$a_2 = a_4 = a_5 = a_7 = a_8 = \cdots = a_{3t-5} = a_{3t-4} = 1.$$

If the remaining nodes can be distributed in such a way that a_1 and a_{3t-3} are at least as large as $\left[\frac{n-1}{t}\right] - 1$, and each of the remaining $t - 2$ integers a_i is positive and at least as large as $\left[\frac{n-1}{t}\right] - 2$, then it is clear that the degree of each node in the resulting connected graph will be at least $\left[\frac{n-1}{t}\right] - 1$. Upon counting the number of nodes needed in the $3t - 1$ classes, we see that these requirements can be satisfied if

$$(2t - 1) + 2\left(\left[\frac{n-1}{t}\right] - 1\right) + (t - 2)\left(\left[\frac{n-1}{t}\right] - 2\right) = t\left[\frac{n-1}{t}\right] + 1 \leq n$$

when $k \leq n - 4$, or if $3t - 1 \leq n$ when $k = n - 3$ or $n - 2$. But these inequalities are certainly true, and hence the lower bound for $g(n, k)$ holds.

The required result for the case $k = 3t - 3$ now follows if we combine these bounds for $g(n, k)$. The proofs for the remaining two cases involve only minor modifications in the above argument, and therefore we omit them.

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