ON THE FIBRE HOMOTOPY TYPE OF NORMAL BUNDLES

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1. INTRODUCTION

It was proved by Atiyah [1] that the fibre homotopy type of the stable normal sphere bundle of a manifold M is an invariant of the homotopy type of M. Theorem A below (discovered before I learned of Atiyah's proof) gives an elementary proof of this fact, and also applies to nonstable cases. (See [5], for example. One purpose of the present work is to supply item 6 in the bibliography of [5].) The situation considered is a homotopy-commutative diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{f} & M_1 \\ & {\scriptstyle g_0}\searrow \quad \nearrow{\scriptstyle g_1} & \\ & U_1 & \end{array}$$

with f a homotopy equivalence, g_i: $M_i \to V$ embeddings (i = 0, 1), and U_1 a closed tubular neighborhood of $g_1(M_1)$. Theorem A implies that the normal sphere bundles of g_0 and g_1 are fibre-homotopically equivalent. Theorem B applies Theorem A to the problem of choosing g_0 (given f and g) so that it will have as many independent normal vector fields as g_1.

The proof of Theorem A in the case $\dim V \geq \dim M + 3$ depends on Lemma 2, due to Milnor, which states that if U_0 is a closed tubular neighborhood of $g_0(M_0)$ inside int U_1, then U_1 - int U_0 is an h-cobordism between the boundaries bU_1 and bU_0. This Lemma is no longer universally true if $\dim V = \dim M + 2$; Theorem C (which is independent from Theorems A and B) exhibits a special case where it is true. An immediate corollary is that *if* $M_0 \times R^k$ *is diffeomorphic to* $M_1 \times R^k$, *then* $M_0 \times S^{k-1}$ *is* h-*cobordant to* $M_1 \times S^{k-1}$. (The interesting case is k = 2.)

All manifolds, immersions, and embeddings are *smooth.*

Throughout the paper, M_0 and M_1 are compact unbounded manifolds of dimension m, and V is a Riemannian manifold of dimension v.

2. FIBRE HOMOTOPY TYPE

If α and β are bundles, then $\alpha \sim \beta$ indicates that α and β are isomorphic, while $\alpha \simeq \beta$ means that α and β have the same *fibre homotopy type.* For this concept, the reader is referred to Dold [3].

Let f: M $\to$ V be an immersion. If ν is the normal vector space bundle of f, then $\hat{\nu}$ will denote the normal sphere bundle of f, and conversely.

THEOREM A. *Let* g_i: $M_i \to V$ *be embeddings* (i = 0, 1). *Let* $U_1 \subset V$ *be a closed tubular neighborhood of* $g_1(M_1)$ *such that* $g_0(M_0) \subset U_1$. *Let* f: $M_0 \to M_1$ *be a homotopy equivalence making a homotopy-commutative diagram*

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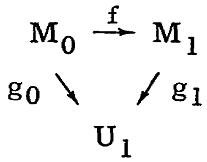


Let $\hat{\nu}_i$ *be the normal sphere bundle of* g_i. *Then*

(a) $f^* \hat{\nu}_1 \simeq \hat{\nu}_0$;

(b) $f^* \nu_1 \sim \nu_0$ *if* $v - m \leq 2$.

Remark. If $v \geq \frac{3}{2}(m + 1)$, and f, g_1 are given, then g_0 always exists, by Haefliger [4].

The proof of Theorem A occupies the rest of this section.

In Lemma 1, all the spaces involved are to be CW-complexes.

LEMMA 1. *Let* $p_i : E_i \to B_i$ *be fibre spaces* $(i = 0, 1)$. *A map* $\phi : E_0 \to E_1$ *is a fibre-homotopy equivalence provided*

(a) ϕ *is a homotopy equivalence,*

(b) ϕ *covers a homotopy equivalence* $B_0 \to B_1$.

Proof. See Dold [3].

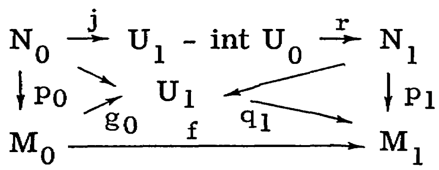
Now let $U_0 \subset \text{int } U_1$ be a closed tubular neighborhood of $g_0(M_0)$. (We may assume $g_0(M_0) \subset \text{int } U_1$.) Put $N_i = bU_i$.

LEMMA 2 (Milnor). *If* $v \geq m + 3$, *then* N_0 *and* N_1 *are deformation retracts of* $U_1 - \text{int } U_0$.

Proof. See Lemma 2 of [6].

LEMMA 3. *If* N_0 *and* N_1 *are deformation retracts of* $U_1 - \text{int } U_0$, *then* $f^* \hat{\nu}_1 \simeq \hat{\nu}_0$.

Proof. Consider the diagram



The maps are defined as follows:

(i) Unlabelled maps are inclusions, and so is j.

(ii) Identify U_i with the normal disk bundle of g_i. Then $q_i : U_i \to M_i$ is the bundle projection, and $p_i = q_i \mid N_i$ is the bundle projection of the normal sphere bundle $\hat{\nu}_i$.

(iii) r is a homotopy inverse to the inclusion $N_1 \to U_1 - \text{int } U_0$, which exists by hypothesis.

Each triangle is homotopy-commutative.

The covering homotopy property of the fibre space $p_1 : N_1 \to M_1$ implies that $rj : N_0 \to N_1$ is homotopic to a map covering $f : M_0 \to M_1$. By hypothesis rj is a homotopy equivalence; consequently Lemma 1 implies Lemma 2.

Part (a) of Theorem A is proved by the lemmas. To prove part (b), first observe that

$$\text{(iv)} \qquad f^{\#} w_1(\nu_1) = w_1(\nu_0);$$

$f^{\#}$ is the induced homomorphism of cohomology, and w_1 denotes the first Stiefel-Whitney class. This follows from the homotopy commutativity of the diagram in Theorem A, the Whitney sum theorem, and the fact that $f^{\#} w_1(M_1) = w_1(M_0)$. If $v = m + 1$, then (iv) suffices to prove that $f^* \nu_1 \sim \nu_0$.

There remains the case $v = m + 2$.

Let Γ be a local system of groups on a space Y. For each map $f: X \to Y$ there is an induced local system $f^{-1}\Gamma$ on X. Let

$$f_{\#}: H_i(X; f^{-1}\Gamma) \to H_i(Y; \Gamma) \qquad \text{and} \qquad f^{\#}: H^i(Y; \Gamma) \to H^i(X; f^{-1}\Gamma)$$

be the induced homomorphisms. If f is the inclusion of the open subset X into Y, and $\mathscr{H}$ denotes homology based on infinite chains, there exists an induced homomorphism $f^! : \mathscr{H}_i(Y; \Gamma) \to \mathscr{H}_i(X; f^{-1}\Gamma)$.

Let F_i be the local system on M_i determined by orientations of the fibres of ν_i. As was shown earlier, $f^{\#} w_1(\nu_1) = w_1(\nu_0)$. Equivalently, $f^{-1} F_1 = F_0$. It is known that $f^* \nu_1 \sim \nu_0$ provided that $f^{\#} X_1 = X_0$, where $X_i \in H^2(M_i; F_i)$ is the Euler class of ν_i. We proceed to prove the last equality.

Let $V_i = \operatorname{int} U_i$. Since $f_i: M_i \to V_i$ is a homotopy equivalence, there exist unique local systems $\tilde{F}_i$ on V_i such that $f_i^{-1} \tilde{F}_i = F_i$. Let G_i be the local system corresponding to orientation of the tangent planes of M_i, and let $\tilde{G}_i$ be the local system on V_i such that $f_i^{-1} \tilde{G}_i = G_i$. Since $g_0 \simeq f g_1$ in V_1, it follows that if $\phi: U_0 \to U_1$ is the inclusion, then $\phi^{-1} G_1 = G_0$.

The local system of orientations of V_i is easily seen to be $\tilde{F}_i \otimes \tilde{G}_i$. Therefore there is an isomorphism of Poincaré duality $D_i: \mathscr{H}_m(V_i; \tilde{G}_i) \to H^2(V_i; \tilde{F}_i)$; see [2, p. 4].

It is known that the following diagram commutes:

$$\begin{array}{ccccccc}
H_m(M_1; G_1) & \xrightarrow{g_{1\#}} & \mathscr{H}_m(V_1; \tilde{G}_1) & \xrightarrow{D_1} & H^2(V_1; \tilde{F}_1) & \xrightarrow{g_1^{\#}} & H^2(M_1; F_1) \\
\uparrow f_{\#} & & \downarrow \phi^! & & \downarrow \phi^{\#} & & \downarrow f^{\#} \\
H_m(M_0; G_0) & \xrightarrow{g_{0\#}} & \mathscr{H}_m(V_0; \tilde{G}_0) & \xrightarrow{D_0} & H^2(V_0; \tilde{F}_0) & \xrightarrow{g_0^{\#}} & H^2(M_0; F_0)
\end{array}$$

Let $m_i \in H_m(M_i; G_i)$ be generators such that $f_{\#}(m_0) = m_1$. A theorem of Thom [7] states that $g_i^{\#} D_i g_{i\#}(m_0) = X_i$. The commutativity of the diagram shows that $f^{\#} X_1 = X_0$; it follows that $f^* \nu_1 = \nu_0$. Theorem A is proved.

3. NORMAL FRAME FIELDS

Let $g: M \to V$ be an immersion of a manifold in V. We say g is q-*framable* if g admits q linearly independent normal vector fields.

THEOREM B. *Let a positive integer* q *satisfy the condition* $2(v - q) \geq 3m + 1$. *Let* $f: M_0 \to M_1$ *be a homotopy equivalence and* $g_1: M_1 \to V$ *an immersion or embedding that is* q*-framable.*

(a) *If* g_1 *is an embedding and* U *a tubular neighborhood of* $g_1(M_1)$, *there exists a* q*-framable embedding* $g_0: M_0 \to U$ *that is homotopic to* $g_1 f$ *in* U.

(b) *If* g_1 *is an immersion, there exists an immersion* $g_0: M_0 \to V$ *that is homotopic to* $g_1 f$.

(c) *In both* (a) *and* (b), $f^* \hat{\nu}_1 \simeq \hat{\nu}_0$, *where* $\hat{\nu}_i$ *is the normal sphere bundle of* g_i.

Proof. (a) To say that the embedding $g_1: M_1 \to V$ is q-framable means that the normal (v - m)-plane bundle ν_1 is a Whitney sum $\nu_1 = \mu_1 \oplus \varepsilon^{q-1}$, where ε^{q-1} is the trivial (q - 1)-plane bundle, and the (v - m - q + 1)-plane bundle μ_1 has a nonzero section. Factor g_1 thus:

$$M_1 \xrightarrow{h_1} E\mu_1 \subset E\nu_1 \xrightarrow{e} U,$$

where E indicates total space, h_1 is the zero cross-section, and e is a diffeomorphism. The dimension of $E\mu_1$ is v - q + 1. By hypothesis, dim $E\mu_1 \geq 3(m+1)/2$. Therefore the embedding theory of Haefliger [4] is applicable and says that because the map $h_1 f: M_0 \to E\mu_1$ is a homotopy equivalence, it is homotopic to an embedding $h_0: M_0 \to E\mu_1$. Let $\hat{\mu}_0$ be the normal sphere bundle of h_0, and $\hat{\mu}_1$ that of h_1. By Theorem A, $f^* \hat{\mu}_1 \simeq \hat{\mu}_0$. Since $\hat{\mu}_1$ has a section, so has $\hat{\mu}_0$. Clearly the normal (v - m)-plane bundle ν_0 of the composite embedding g_0, defined to be

$$M_0 \xrightarrow{h_0} E\mu_1 \subset E\nu_1 \xrightarrow{e} U,$$

has the form $\alpha \oplus \varepsilon^1 \oplus \varepsilon^{q-1}$. Therefore g_0 is q-framable, and this proves (a).

Part (b) follows from (a), since an immersion $g_1: M_1 \to V$ can be factored: $M_1 \xrightarrow{h_1} E\nu_1 \xrightarrow{e} V$, where ν_1 is the normal (v - m)-plane bundle of g_1, h_1 is an embedding, and e is an immersion.

Part (c) is a consequence of Theorem A.

4. HOMEOMORPHISMS OF BUNDLES

Let E_i be the total space of a smooth orthogonal k-plane bundle over M_i (i = 0, 1). Let $B_i \subset E_i$ be the corresponding unit ball bundle. For each $t > 0$ put $tB_i = \{tx \mid tx \in E_i, x \in B_i\}$; let $N_i = bB_i$ and $tN_i = b(tB_i)$.

THEOREM C. *Let* $\psi: E_0 \to E_1$ *be a diffeomorphism such that* $\psi B_0 \subset \text{int } B_1$. *Then* $B_1 - \text{int } hB_0$ *is an* h*-cobordism between* ψN_0 *and* N_1.

Remark. This theorem actually has little to do with manifolds. It can be reformulated so as to apply to bundles over paracompact spaces, with essentially the same proof.

Proof. By compactness of M_i, we can choose r, s > 0 such that

$$B_1 \subset \text{int } \psi(rB_0) \quad \text{and} \quad \psi(rB_0) \subset \text{int } sB_1.$$

(i) $B_1 - \text{int}\, \psi B_0$ is diffeomorphic to $s B_1 - \text{int}\, \psi(r B_0)$.

To prove (i), choose $t < 1$ so that $\psi B_0 \subset \text{int}\, t B_1$. Let $\phi: E_1 \to E_1$ be a diffeomorphism taking B_1 onto $s B_1$, leaving $t B_1$ fixed. Then choose $u > r$ so that $\psi(u B_0) \subset \text{int}\, s B_1$, and let $\theta: E_0 \to E_0$ be a diffeomorphism taking B_0 onto $r B_0$, leaving $E_0 - \text{int}\, u B_0$ fixed. Then $\psi \theta \psi^{-1} \phi: E_1 \to E_1$ is a diffeomorphism taking $B_1 - \text{int}\, \psi B_0$ onto $s B_1 - \text{int}\, \psi(r B_0)$.

(ii) ψN_0 is a deformation retract of $B_1 - \text{int}\, \psi B_0$.

This follows easily from the facts that ψN_0 is a deformation retract of $\psi r B_0 - \text{int}\, \psi B_0$, and B_1 is a retract of $s B_1$. Similarly,

(iii) $s N_1$ is a deformation retract of $s B_1 - \text{int}\, \psi(r B_0)$.

From (i) and (iii) it follows that N_1 is a deformation retract of $B_1 - \text{int}\, \psi B_0$. This together with (ii) proves Theorem C.

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