# STIRLING SUMMABILITY OF RAPIDLY DIVERGENT SERIES

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## 1. INTRODUCTION

A summability method based on the Stirling numbers and a parameter $\lambda$ was introduced by Karamata [5], who called it the Stirling method and denoted it by $S(\lambda)$. We shall use $\mathscr{S}(\lambda)$ for a slight modification of this method. The special case $\lambda = 1$ of $\mathscr{S}(\lambda)$ was studied independently by Lototsky [7] and developed by Agnew [1], [2], who named this case the Lototsky method, denoted by L. To illustrate the power of the method, Agnew showed that Euler's series $\Sigma(-1)^k k!\, z^{-k}$ is L-summable if (with $z = x + iy$) $x \geq \log 2$, but not if $|z| < \log 2$; the intermediate region remained in doubt [1, p. 111]. The purpose of the present note is to present a general theorem on Stirling summability, which will show in particular that Euler's series is L-summable if z is outside the first arch of $x = \log(2 \cos y)$, but not if z is inside. By a separate argument, we can show that the series is summable on the boundary also. Furthermore, for $\mathscr{S}(\lambda)$-summability $(\lambda > 0)$, we obtain the same region multiplied by $\lambda$; therefore the series is summable by some member of the family in the whole plane, except on the negative real axis.

It was pointed out by the referee that Greub [4] used the same curve $x = \log(2 \cos y)$ for somewhat similar purposes. Greub's paper appeared almost simultaneously with [2], and it reached the same conclusions about the relations among the Lototsky and other summability methods.

## 2. DEFINITIONS

We define the Stirling numbers $p_{nk}$ $(n = 1, 2, \cdots;\ k = 0, 1, 2, \cdots, n)$ by the identity

$$x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=0}^{n} p_{nk} x^k ;$$

thus $p_{n0} = 0$ $(n = 1, 2, \cdots)$, and we define also $p_{00} = 0$. The Stirling method was defined by Karamata by the formula

$$S(\lambda): \quad \sigma_n = \frac{1}{(\lambda)_n} \sum_{k=0}^{n} p_{nk} \lambda^k s_k ,$$

where $(\lambda)_n = \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)$; if $\sigma_n \to \sigma$ as $n \to \infty$, we say the sequence $\{s_0, s_1, s_2, \cdots\}$ is $S(\lambda)$-limitable to $\sigma$. We always assume that $\lambda > 0$, which ensures regularity.

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We shall modify the method slightly, applying it to the sequence $\{s_1, s_2, \cdots\}$ by writing

$$\mathscr{S}(\lambda): \quad \sigma_n = \frac{1}{(\lambda)_n} \sum_{k=1}^{n} p_{nk} \lambda^k s_k,$$

in order that the special case $\lambda = 1$ may agree exactly with the Lototsky method L as written by Lototsky and Agnew. (Precision is in order, since the method is not translative.)

It was shown in [1, p. 114-115] that the series-to-series form of L is

$$U_1 = u_1,$$

$$U_n = \frac{1}{n!} \sum_{k=1}^{n} p_{n-1,k} u_{k+1} \qquad (n = 2, 3, \cdots);$$

if $\Sigma U_n = \sigma$, we say the series $\Sigma u_k$ is L-summable to $\sigma$. For our purposes it will be more convenient to renumber the terms of the series, starting from $u_0$ and $U_0$, to obtain [2, p. 364]

$$U_0 = u_0,$$

$$U_n = \frac{1}{(n+1)!} \sum_{k=1}^{n} p_{nk} u_k \qquad (n = 1, 2, \cdots).$$

By the method used in [1], we find that the series-to-series form of $\mathscr{S}(\lambda)$ is

(1)
$$U_0 = u_0,$$
$$U_n = \frac{1}{(\lambda+1)_n} \sum_{k=1}^{n} p_{nk} \lambda^k u_k \qquad (n = 1, 2, \cdots).$$

Now it is well known that if we take the branch of $-\log(1 - w)$ defined by $\Sigma w^n/n$, then

$$\{-\log(1 - w)\}^k = k! \sum_{n=k}^{\infty} \frac{p_{nk}}{n!} w^n \qquad (k = 1, 2, \cdots).$$

This is easily proved inductively: we differentiate to obtain a relation between $\{-\log(1 - w)\}^{k+1}$ and $\{-\log(1 - w)\}^k$, and use the recursion formula

$$p_{n+1,k} = n p_{n,k} + p_{n,k-1}.$$

It follows at once that

$$\{-\lambda \log(1 - w)\}^k = k! \sum_{n=k}^{\infty} \frac{p_{nk} \lambda^k}{n!} w^n \qquad (k = 1, 2, \cdots).$$

We may therefore regard the $\mathscr{S}(\lambda)$-method as generated in the following way. Given the series $\Sigma_{k=0}^{\infty} u_k$, write the series

$$f(t) = \sum_{k=0}^{\infty} \frac{u_k}{k!} t^k,$$

put $t = -\lambda \log(1 - w)$, and arrange $F_\lambda(w) = f(-\lambda \log(1 - w))$ in powers of $w$:

$$F_\lambda(w) = u_0 + \sum_{k=1}^{\infty} u_k \sum_{n=k}^{\infty} \frac{p_{nk}\lambda^k}{n!} w^n$$

$$= u_0 + \sum_{n=1}^{\infty} w^n \sum_{k=1}^{n} \frac{p_{nk}\lambda^k}{n!} u_k.$$

Denoting $F_\lambda(w)$ by $\Sigma_{n=0}^{\infty} h_{\lambda,n} w^n$, we see by (1) that

$$U_n = \frac{n!}{(\lambda+1)_n} h_{\lambda,n}, \tag{2}$$

and we may hope to determine whether $\Sigma U_n$ converges by considering the properties of $F_\lambda(w)$.

We note in passing that it is easily seen by classical analysis that if the series $\Sigma U_n w^n$ has a positive radius of convergence, then the same is true of $\Sigma (u_k/k!) t^k$, as was stated by Agnew [2, p. 366].

## 3. THE MAIN RESULT

THEOREM. *Given a series*

$$\sum_{k=0}^{\infty} \frac{a_k}{z^k}, \tag{3}$$

*let*

$$f_z(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{t}{z}\right)^k$$

*and*

$$F_{z,\lambda}(w) = f_z(-\lambda \log(1 - w)) = \sum h_n w^n$$

*(where the* $h_n$ *depend on* $z$ *and* $\lambda$*). Then*

(i) *for all* $z$ *such that* $F_{z,\lambda}$ *is regular and bounded in* $|w| < 1$, *the series (3) is* $\mathscr{S}(\lambda)$*-summable;*

(ii) *for all* $z$ *such that* $F_{z,\lambda}$ *has a singularity in* $|w| < 1$, *the series (3) is not* $\mathscr{S}(\lambda)$*-summable;*

(iii) *for values* z *such that* $F_{z,\lambda}$ *is regular but not bounded in* $|w| < 1$, *the series (3) may or may not be* $\mathscr{S}(\lambda)$*-summable.*

*Proof.* (i) It was shown by Landau [6, p. 446] that the factorial series

$$\sum U_n = U_0 + \sum_{n=1}^{\infty} \frac{n!\, h_n}{(\lambda+1)_n},$$

based on (2), converges if and only if the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{h_n}{n^\lambda} \tag{4}$$

converges. Also, if $S_n = h_1 + \cdots + h_n$, then

$$\sum_{n=1}^{p} \frac{h_n}{n^\lambda} = \sum_{n=1}^{p-1} S_n \left(\frac{1}{n^\lambda} - \frac{1}{(n+1)^\lambda}\right) + \frac{S_p}{p^\lambda}.$$

Now

$$\frac{1}{n^\lambda} - \frac{1}{(n+1)^\lambda} = O\left(\frac{1}{n^{\lambda+1}}\right),$$

and under our assumption that $F(w)$ is regular and bounded on $|w| < 1$, it follows from another result of Landau's [3, pp. 442-443] that $S_n = O(\log n)$ as $n \to \infty$ for fixed $\lambda$, z. Hence (4) converges, and this completes the proof of part (i).

(ii) This part is obvious.

(iii) As an example we consider the L-summation of the geometric series $\Sigma \zeta^n$. Here

$$f(t) = e^{\zeta t}, \qquad F(w) = e^{-\zeta \log(1-w)}.$$

The image of $|w| = 1$ in the t-plane $(t = \alpha + i\beta)$ is the first arch of

$$\alpha = -\log(2\cos\beta),$$

which extends to infinity in the positive real direction, in a strip of width $\pi$, the unit disk being mapped into the inside of the arch. If $\Re\zeta < 0$, then $\Re(-\zeta\log(1-w))$ is bounded above, and $|F(w)|$ is bounded, on $|w| < 1$; by part (i), $\Sigma\zeta^n$ is L-summable. If $\Re\zeta > 0$, we no longer have the boundedness; nevertheless we know by [1, p. 107] that $\Sigma\zeta^n$ is L-summable if $\Re\zeta < 1$, but not if $\Re\zeta \geq 1$.

## 4. APPLICATIONS

We consider first the series $\Sigma_{k=0}^{\infty} (-1)^k k!\, z^{-k}$ mentioned in the Introduction. Here

$$f_z(t) = \frac{z}{z+t},$$

$$F_{z,\lambda}(w) = \frac{z}{z - \lambda \log(1 - w)}.$$

It is a question of whether $z$ is or is not a value taken by $\lambda \log(1 - w)$ in the unit circle. More precisely, the image of $|w| = 1$ by $z = \lambda \log(1 - w)$ is the first arch of

$$\lambda^{-1} x = \log(2 \cos \lambda^{-1} y),$$

which we denote by $C_\lambda$, and parts (i), (ii), (iii) of the theorem apply according as $z$ is outside, inside, or on $C_\lambda$; therefore the series is $\mathscr{S}(\lambda)$-summable if $z$ is outside $C_\lambda$, but not if $z$ is inside. The curve $C_\lambda$ is of course the curve $C_1$: $x = \log(2 \cos y)$, multiplied by $\lambda$, and as $\lambda \to 0$, the curves $C_\lambda$ approach the negative real axis. As $\lambda$ decreases, the methods $\mathscr{S}(\lambda)$ become stronger, with consistency [8]; since the transformed series converges uniformly in each closed bounded region outside $C_\lambda$, the $\mathscr{S}(\lambda)$-sum of the series is the analytic continuation of the L-sum outside $C_1$, namely, the Borel value

$$\int_0^\infty \frac{ze^{-t}}{z + t}\, dt,$$

in the cut plane. (The generalized Borel method includes L; see [1, Section 11].)

It was pointed out by M. Wyman, in correspondence, that for each $\lambda > 0$ the series $\Sigma(-1)^k k!\, z^{-k}$ is $\mathscr{S}(\lambda)$-summable on the curve $C_\lambda$ itself. This may be proved from Cauchy's integral:

$$h_n = \frac{1}{2\pi i} \int_C \frac{1}{w^{n+1}} \frac{z}{z - \lambda \log(1 - w)}\, dw.$$

We observe that if $z$ is on $C_\lambda$, the integrand has a branch point at $w = 1$ and a simple pole at the point $w = 1 - e^{z/\lambda}$ on the unit circle. If we take $C$ to be a circle of radius $R > 1$, with keyholes coming in to the branch point and the pole, we can show by straightforward estimates that $\Sigma h_n/n^\lambda$ converges.

We next consider the series $\Sigma_{k=1}^\infty (B_k/k) z^{-k}$, the $B_k$ denoting the Bernoulli numbers. The case $z = 1$ was treated in [1, Section 10]. We set

$$u_k = B_{k+1} (k + 1)^{-1} z^{-k-1},$$

and for simplicity we consider only $\lambda = 1$. We find

$$f_z(t) = \frac{1}{z} \frac{1}{e^{t/z} - 1} - \frac{1}{t},$$

$$F_{z,1}(w) = \frac{1}{z} \frac{1}{e^{(-\log(1-w))/z} - 1} + \frac{1}{\log(1 - w)}.$$

Thus $F_{z,1}$ is bounded in $|w| < 1$ provided the equation $\log(1 - w) = 2\pi n i z$ has no roots in $|w| < 1$ for any $n = \pm 1, \pm 2, \cdots$. The image of $|w| = 1$ by

$$\log(1 - w) = 2\pi n i z$$

is the curve $C_1$ of the preceding example, divided by $2\pi ni$; this encloses the positive or negative imaginary axis according as n is positive or negative. For L-summability it is sufficient that z be excluded from the regions corresponding to $n = 1$ and $n = -1$; these contain the whole imaginary axis. For the allowable values of z, the L-sum of the series is the Borel value

$$\int_0^\infty e^{-t}\left(\frac{1}{z}\frac{1}{e^{t/z}-1} - \frac{1}{t}\right)dt = \log z - \frac{\Gamma'(1+z)}{\Gamma(1+z)}.$$

We consider finally the series

$$1 + 0 - \frac{2!}{1!\,z^2} + 0 + \frac{4!}{2!\,z^4} + \cdots.$$

Taking again $\lambda = 1$, we find that

$$f_z(t) = e^{-(t/z)^2}, \qquad F_{z,1}(w) = e^{-(\log(1-w))^2/z^2}.$$

The domain of summability consists of the quadrant $|\arg z| < \pi/4$ and the opposite quadrant.

These three examples show that while the method provided by the theorem is fairly general, the results of applying it in special cases may vary considerably.

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