LENGTH AND AREA ESTIMATES FOR ANALYTIC FUNCTIONS

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Suppose that $f(z)$ is analytic for $|z| < 1$, and for $0 < r \leq 1$ let $D(r, f)$ denote the image of $|z| < r$ under $f(z)$. Let $A(r, f)$ denote the area of $D(r, f)$, in the sense that the areas of regions of $D(r, f)$ covered more than once are counted with the appropriate multiplicity. Also, let $L(r, f)$ denote the length of the curve $w = f(re^{i\theta})$ $(0 \leq \theta \leq 2\pi)$. Then, for $r < 1$,

$$A(r, f) \geq \pi r^2 |f'(0)|^2 , \tag{1}$$

$$L(r, f) \geq 2\pi r |f'(0)| , \tag{2}$$

and in either estimate equality holds only for the functions $f(z) = a_0 + a_1 z$ [3, p. 155, Problems 10, 11].

We improve (1) and (2) in the following way. If $a(r, f)$ denotes the area of the set $D(r, f)$, then evidently $A(r, f) \geq a(r, f)$. Theorem 1 (with $n = 1$) asserts that $a(r, f) \geq \pi r^2 |f'(0)|^2$. This estimate is precise only for the functions for which (1) is precise.

Let $C(r, f)$ be the outer boundary of $D(r, f)$, and let $\ell(r, f)$ be the length of $C(r, f)$. Here we mean length to be measured as seen; that is, the lengths of multiply covered arcs of $C(r, f)$ are counted only once. Then $L(r, f) \geq \ell(r, f)$, and Theorem 2 (with $n = 1$) asserts that $\ell(r, f) \geq 2\pi r |f'(0)|$. Again, equality occurs only for the functions $f(z) = a_0 + a_1 z$. If $\ell^*(r, f)$ denotes the length of the boundary of $D(r, f)$ (in the above sense) then $L(r, f) \geq \ell^*(r, f) \geq \ell(r, f)$, so that, in particular, our result implies that $\ell^*(r, f) \geq 2\pi r |f'(0)|$.

If $A(1, f)$ exists, then (1) also holds for $r = 1$. Indeed, if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n , \tag{3}$$

then the formula

$$A(r, f) = \pi \sum_{n=1}^{\infty} n |a_n|^2 r^{2n} \tag{4}$$

is valid even for $r = 1$, in the case where $A(1, f)$ is finite.

If $f(z)$ is continuous for $|z| \leq 1$ and of bounded variation on $|z| = 1$, then (2) is valid for $r = 1$. This follows since (2) holds for all $r < 1$, and with these additional hypotheses $L(r, f) \to L(1, f)$ as $r \to 1$ [5, p. 150, 6.11].

If $f(z)$ has the form (3), then (4) implies that

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$$A(r, f) \geq \pi n r^{2n} |a_n|^2. \tag{5}$$

Also, from the relations

$$na_n = \frac{1}{2\pi i} \int_{|z|=r} \frac{f'(z)}{z^n} dz, \qquad L(r, f) = \int_0^{2\pi} |f'(re^{i\theta})| \, r \, d\theta$$

one readily obtains

$$L(r, f) \geq 2\pi r^n n |a_n|. \tag{6}$$

Inequalities (5) and (6) are the same as (1) and (2) if $n = 1$, and they are precise only for the functions $f(z) = a_0 + a_n z^n$, which map $|z| < r$ onto a circle covered n times (if $a_n \neq 0$). If the power series for $f(z)$ begins $f(z) = a_0 + a_n z^n + a_{n+1} z^{n+1} + \cdots$, then the factor n in (5) and (6) can be replaced by 1 when $a(r, f)$ and $\ell(r, f)$ are considered; that is, in this case

$$a(r, f) \geq \pi r^{2n} |a_n|^2, \tag{7}$$

$$\ell(r, f) \geq 2\pi r^n |a_n|. \tag{8}$$

For $n = 1$, (7) and (8) are the results announced earlier.

The proof of (7) depends upon an application of the principle of symmetrization [1, Chapter 4], [4], whereas (8) is established by means of the principle of subordination [3, p. 226]. It is interesting to note that the method of proving (8) can be used to obtain a weaker form of (7), and that the method of proving (7) can be used to obtain a weaker form of (8), namely $\ell^*(r, f) \geq 2\pi r^n |a_n|$.

THEOREM 1. *Let* $f(z) = a_0 + a_n z^n + a_{n+1} z^{n+1} + \cdots$ *be analytic for* $|z| < 1$, *and for* $0 < r < 1$ *let* $D(r, f)$ *denote the image of* $|z| < r$ *under* $f(z)$. *If* $a(r, f)$ *is the area of* $D(r, f)$, *then* $a(r, f) \geq \pi r^{2n} |a_n|^2$.

Proof. The function $g(z) = f(rz) = b_0 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ is analytic for $|z| < 1$ and maps $|z| < 1$ onto $D(r, f)$. Because of the form of $g(z)$,

$$|b_n| \leq r_1, \tag{9}$$

where r_1 is the inner radius of $D(r, f)$ at b_0. This can be proved by the method used for the case $n = 1$ [1, p. 80], except that the function $h(z)$ in that argument must be replaced by

$$H(z) = g_\rho(g(z), b_0) + \log \frac{|z|^n}{\rho^n},$$

so that $H(0) = \log[r(\rho)/|b_n|\rho^n] \geq 0$. (The notation $g_\rho(w, b_0)$ is the same as that used in [1].)

Let D be the Steiner symmetrization of $D(r, f)$ with respect to any line through b_0. Since D is a bounded simply connected domain, there exists a function $h(z) = c_0 + c_1 z + \cdots$, analytic and univalent for $|z| < 1$, mapping $|z| < 1$ onto D, and such that $c_0 = b_0$. Now $r_1 \leq r_2 = |c_1|$, where r_2 is the inner radius of D with respect to c_0. Together with (9), this implies that

$$|b_n| \leq |c_1|. \tag{10}$$

The area of D (call it A) exists, since $h(z)$ is univalent and bounded, and

$$A \geq \pi |c_1|^2. \tag{11}$$

Since Steiner symmetrization preserves area, $A = a(r, f)$, and from (11), (10), and the relation $b_n = r^n a_n$ we obtain the result $a(r, f) \geq \pi r^{2n} |a_n|^2$.

This theorem is exact, since $a(r, f) = \pi r^{2n} |a_n|^2$ for the functions

$$f(z) = a_0 + a_n z^n.$$

For $n = 1$, $a(r, f) = \pi r^2 |a_1|^2$ only for the functions $f(z) = a_0 + a_1 z$. To prove this, notice that $a(r, f) = \pi r^2 |a_1|^2$ demands $A = \pi |c_1|^2$ and $|b_1| = |c_1|$. The condition $A = \pi |c_1|^2$ implies that $h(z) = c_0 + c_1 z$, and equality in the principle of symmetrization implies that $g(z) = h(\varepsilon z)$, where $|\varepsilon| = 1$. Since $g(z) = f(rz)$, this shows that $f(z) = a_0 + a_1 z$.

THEOREM 2. *Let* $f(z) = a_0 + a_n z^n + a_{n+1} z^{n+1} + \cdots$ *be analytic for* $|z| < 1$, *and for* $0 < r < 1$ *let* $D(r, f)$ *denote the image of* $|z| < r$ *under* $f(z)$. *If* $\ell(r, f)$ *is the length of the outer boundary of* $D(r, f)$, *then* $\ell(r, f) \geq 2\pi r^n |a_n|$.

Proof. Let E be the unbounded component of the complement of $D(r, f)$, and let F be the complement of E. Then F consists of $D(r, f)$ and the "holes" in $D(r, f)$. The outer boundary of $D(r, f)$, which we denote by C, is defined as the boundary of E, or equivalently as the boundary of F. Thus C consists of a finite number of analytic arcs.

There is no loss of generality in assuming that $f(z)$ is nonconstant, in which case F is a simply connected domain [2, Lemma 2]. F is also bounded, and therefore there exists a function $h(z) = c_0 + c_1 z + \cdots$, analytic and univalent for $|z| < 1$, mapping $|z| < 1$ onto F, and such that $c_0 = a_0$. Since the function

$$g(z) = f(rz) = b_0 + b_n z^n + b_{n+1} z^{n+1} + \cdots$$

maps $|z| < 1$ onto $D(r, f)$, and since $b_0 = a_0$ and $D(r, f) \subset F$, the function $g(z)$ is subordinate to $h(z)$. Thus there exists a function $\phi(z) = d_1 z + d_2 z^2 + \cdots$ that is analytic and satisfies $|\phi(z)| < 1$ for $|z| < 1$, and such that $g(z) = h(\phi(z))$. Because of the form of the power series for $g(z)$, and since $g(z) = h(\phi(z))$ and $c_1 \neq 0$, one finds successively that $d_1 = 0$, $d_2 = 0$, $\cdots$, $d_{n-1} = 0$, and $b_n = c_1 d_n$. Using $|d_n| \leq 1$, we obtain

$$|b_n| \leq |c_1|. \tag{12}$$

Now $h(z)$ can be defined continuously for $|z| \leq 1$, and $h(z)$ is of bounded variation on $|z| = 1$. Moreover, as z traverses the circle $|z| = 1$ once, $h(z)$ traverses C once, always moving in the same direction. Therefore, if L denotes the length of the curve $w = h(e^{i\theta})$ $(0 \leq \theta \leq 2\pi)$, then

$$L = \ell(r, f), \tag{13}$$

$$L \geq 2\pi |c_1|. \tag{14}$$

Our result $\ell(r, f) \geq 2\pi r^n |a_n|$ follows from (12), (13), and (14), since $b_n = r^n a_n$.

To determine when $\ell(r, f) = 2\pi r^n |a_n|$, notice that this demands $|b_n| = |c_1|$ and $L = 2\pi |c_1|$. The condition $|b_n| = |c_1|$ is equivalent to $|d_n| = 1$, which holds only if $\phi(z) = \varepsilon z^n$, where $|\varepsilon| = 1$. Since $f(rz) = h(\phi(z))$, this implies that $h(z)$ is analytic for $|z| \leq 1$, and hence $L = 2\pi |c_1|$ can occur only if $h(z) = c_0 + c_1 z$. Then the relation $\phi(z) = \varepsilon z^n$ shows that $f(z) = a_0 + a_n z^n$. Conversely, for the functions $f(z) = a_0 + a_n z^n$, $\ell(r, f) = 2\pi r^n |a_n|$.

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