STATIONARY SETS FOR CERTAIN CLASSES OF DERIVATES OF DARBOUX FUNCTIONS

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1. INTRODUCTION

The basic theorem of elementary calculus which states that a differentiable function f whose derivative vanishes on an interval [a, b] must be constant on [a, b] has been generalized in several ways. Thus, the hypothesis of differentiability has been weakened (for example, to continuity or approximate continuity), the derivative has been generalized (for example, to approximate derivative, or to a Dini derivative), and the vanishing of f' everywhere has been weakened (for example, to vanishing except possibly on a denumerable set).

The authors of a recent series of interesting articles ([2], [6], [7], [8]) have considered the following question. Let $\mathscr{C}$ be a class of functions defined on [a, b]. A set $E \subset [a, b]$ is called a *stationary set* for $\mathscr{C}$ provided every function in $\mathscr{C}$ that is constant on E is constant on [a, b]. Problem: to determine the class of stationary sets for $\mathscr{C}$. Now, if $\mathscr{C}$ is the class of derivatives (or approximate derivatives or Dini derivatives) of differentiable (or continuous or approximately continuous) functions, then a determination of the class of stationary sets of $\mathscr{C}$ yields as a by-product a generalization of the basic theorem mentioned at the outset.

In this article we prove three theorems that generalize the basic theorem in the manner described. One of these theorems deals with approximate derivatives, one with Dini derivatives, and one with extreme bilateral derivatives. We weaken the differentiability of f to the Darboux property, and our exceptional sets are taken to be totally imperfect. We then interpret our results in terms of the language of stationary sets, and show (in two of the three cases) that our results are the best possible in the sense that the exceptional sets *must* be totally imperfect.

2. PRELIMINARIES

In this section we develop notation and terminology, and we state some of the known results to which we shall refer.

Throughout this article, f denotes a finite function defined on a closed interval $I_0 = [a, b]$. If A is a subset of I_0, then $\sim A$ denotes its complement (relative to I_0), $m^*(A)$ its Lebesgue outer measure, and $m(A)$ its Lebesgue measure (if A is Lebesgue measurable). We also use the symbol $A \sim B$ to denote a set-theoretic difference.

For each function f, we denote by D^+f the upper Dini derivative on the right, by $\bar{f}'$ the upper bilateral derivative, and by f'_{ap} the approximate derivative, provided this approximate derivative exists, finite or infinite. By f(A) we shall mean $\{y: y = f(a) \text{ for some } a \in A\}$. If for every interval $I \subset I_0$ the set f(I) is connected, we say f has the *Darboux property* or f is a *Darboux function.*

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For a development of the theory of approximate derivatives see Saks [9]. An elegant treatment of some of the properties of approximate derivatives can be found in Goffman and Neugebauer [4]. The theorems proved in [4], however, pertain to *finite* approximate derivatives, and are not valid for approximate derivatives, which may be infinite. (See Zahorski [12].)

A word about totally imperfect sets is in order. A set is called totally imperfect if it contains no nonempty perfect subsets. Since a nonempty perfect set must be nondenumerable, it follows that any denumerable set is totally imperfect. There exist, however, nondenumerable totally imperfect sets. For a proof of the existence of such sets see Goffman [3, p. 146]. Now, according to a result of Souslin (see Sierpiński [10, p. 224]) every nondenumerable analytic set contains a nonempty perfect set. In particular, a nondenumerable Borel set cannot be totally imperfect. We use this fact in the proofs of the theorems below to prove that our exceptional sets, which by hypothesis are totally imperfect, are in fact denumerable. (Actually, of course, the *conclusions* of our theorems guarantee that these exceptional sets are *empty.*)

3. MAIN RESULTS

We now proceed to the theorems mentioned in the Introduction. Actually, we prove slightly more than was stated there. We assume in each case that the appropriate derivate of the Darboux function f is equal to a constant λ, except possibly on a totally imperfect set, and conclude that f is of the form $f(x) = \lambda x +$ constant on I_0. For $\lambda = 0$, we have the theorems mentioned in the Introduction. Observe that the case $\lambda \neq 0$ does not follow from the case $\lambda = 0$ as easily as one might expect, because it is not immediately clear that the function g given by $g(x) = f(x) - \lambda x$ has the Darboux property.

In the sequel we make free use of several theorems involving such notions as functions of generalized bounded variation, functions with generalized absolute continuity and functions that satisfy Lusin's condition (N). The relevant definitions, theorems, and notations may be found in Saks [9], particularly in Chapters VII and IX.

THEOREM 1. *Let* f *be a Darboux function whose approximate derivative* f'_{ap} *exists, finite or infinite, at each point of* I_0. *If* $f'_{ap} = \lambda$ *on a set* E *whose complement is totally imperfect, then* f *is of the form* $f(x) = \lambda x +$ constant.

Proof. Let $H = \{x\colon f'_{ap}(x) \neq \lambda\}$. The set H is of type $G_{\delta\sigma}$, because f'_{ap}, being an approximate derivative, is of Baire class at most 2 [12]. On the other hand, H is a subset of $\sim E$ and is therefore totally imperfect. It follows that H is at most denumerable. In particular, $f'_{ap} < \infty$, except possibly on a denumerable set. Thus, there exists a denumerable collection of sets $A_1, A_2, \cdots$ such that $I_0 = \bigcup A_k$ and $f \in BV$ on each of the sets A_k. In addition, f satisfies Lusin's condition (N) on I_0. It is easy to verify that the sets A_k may be taken to be measurable and pairwise disjoint.

Let A be any one of the sets A_k. Let f_A be an extension of $f \mid A$ (the restriction of f to A) to all of I_0 such that f_A is of bounded variation. Then $f'_A = f'_{ap} = \lambda$ on A, except for a set $B \subset A$ with $m(B) = 0$. Since f satisfies Lusin's condition (N), $m(f(B)) = 0$. Furthermore,

$$m^*(f(A \sim B)) = m^*(f_A(A \sim B)) \leq \int_{A\sim B} |f'_A| = |\lambda|\, m(A \sim B) = |\lambda|\, m(A).$$

The inequality in this chain is a consequence of the differentiability of f_A on the set $A \sim B$. It follows that $m^*(f(A)) \leq |\lambda|\, m(A)$. Recalling that A was taken to be any of the sets A_k, and that these sets are measurable and pairwise disjoint, we conclude that

$$m(f(I_0)) \leq \sum m^*(f(A_k)) \leq |\lambda| \sum m(A_k) = |\lambda|\, m(I_0).$$

Since we can apply this argument to any interval $I \subset I_0$, we obtain the inequality $m(f(I)) \leq |\lambda|\, m(I)$. Now, by hypothesis, f possesses the Darboux property, hence $f(I)$ is an interval (possibly degenerate). From this and the inequality

$$m(f(I)) \leq |\lambda|\, m(I)$$

it follows readily that f satisfies a Lipschitz condition with constant $|\lambda|$. In particular, f is absolutely continuous. Since $f' = \lambda$ a.e., we conclude that f is of the form $f(x) = \lambda x + \text{constant}$.

THEOREM 2. *Let* f *be a Darboux-Baire function on* I_0. *If on a set* E *whose complement in* I_0 *is totally imperfect, at least one of the Dini derivatives equals the constant* λ, *then* f *is of the form* $f(x) = \lambda x + \text{constant}$.

Proof. Let H be the set of points at which none of the Dini derivatives equals λ. Then $H \subset \sim E$, hence H is totally imperfect. On the other hand, H is the intersection of four sets: $H = H^+ \cap H_+ \cap H^- \cap H_-$, where for example, H^+ is the set on which $D^+f \neq \lambda$, and where the sets H_+, H^- and H_- are defined in an analogous manner with respect to the other Dini derivatives. Since f is a Baire function, each of its Dini derivatives is a Baire function [1]. It follows that each of the four sets is a Borel set. Thus H is a Borel set, and since H is also totally imperfect, we conclude that it is at most denumerable. From the inequality

$$m^*(f(\sim H^+)) \leq \int_{\sim H^+} |D^+ f| = |\lambda|\, m(\sim H^+) \leq |\lambda|\, m(I_0)$$

and similar inequalities for the sets H_+, H^-, and H_- we deduce that

$$m^*(f(\sim H)) \leq 4\, |\lambda|\, m(I_0).$$

Since $f(H)$ is countable, $m^*(f(I_0)) \leq 4\, |\lambda|\, m(I_0)$. As in the proof of Theorem 1, we conclude that f satisfies a Lipschitz condition and that $f' = \lambda$ a.e.. The theorem now follows.

THEOREM 3. *Let* f *satisfy the Darboux condition on an interval* I_0. *If on a set* E *whose complement is totally imperfect at least one of the extreme bilateral derivatives is equal to the constant* λ, *then* f *is of the form* $f(x) = \lambda x + \text{constant}$.

Proof. According to a theorem of Hájek [5], the extreme bilateral derivatives of an arbitrary finite function are of Baire class at most 2. We use this to show that the set H' on which neither extreme bilateral derivative equals λ is of type $G_{\delta\sigma}$, and we conclude, as before, that H' is at most denumerable. Now, the set H of the previous theorem is contained in H'. The rest of the proof of this theorem parallels the corresponding part of the proof of Theorem 2. We omit the details.

Remark. A theorem similar to Theorem 2 was stated by Sunyer I Balaguer [11]. In that article, however, the author assumed that f is continuous.

4. STATIONARY SETS

The results of Section 3 may be interpreted in terms of the language of stationary sets. Let $\mathscr{A}$ denote the class of approximate derivatives, finite or infinite, of Darboux functions; $\mathscr{D}$ the class of upper right Dini derivatives of Darboux-Baire functions; and $\mathscr{U}$ the class of upper derivatives of Darboux functions.

THEOREM 4. *Let* E *be a subset of* I_0 *whose complement in* I_0 *is totally imperfect. Then* E *is a stationary set for each of the classes* $\mathscr{A}$, $\mathscr{D}$, and $\mathscr{U}$.

The proof of Theorem 4 is contained in Theorems 1, 2, and 3.

Now let E be any subset of I_0 whose complement in I_0 contains a perfect set P'. Let P be a null, nowhere dense, perfect subset of P'. Then there is a continuous, nondecreasing function f that is constant on each interval contiguous to P, but not constant on any interval containing points of P. It is clear that $f' = 0$ on $\sim P$ and therefore on E, but D^+f (and therefore $\bar{f}'$) is positive at all points of P except at the left-hand end points of the intervals contiguous to P. Consequently, any set that is stationary for $\mathscr{D}$ or for $\mathscr{U}$ must have a totally imperfect complement with respect to I_0.

In conclusion we observe that if f is defined by

$$f(x) = \begin{cases} 0 & \text{if } a \leq x < x_0, \\ 1 & \text{if } x = x_0, \\ 2 & \text{if } x_0 < x \leq b, \end{cases}$$

then $f'_{ap}(x) = D^+f(x) = \bar{f}'(x) = 0$ if $x \neq x_0$, but $f'_{ap}(x_0) = D^+f(x_0) = \bar{f}'(x_0) = \infty$.

This simple example shows that none of the theorems of Section 3 is valid without the hypothesis that f be a Darboux function. In fact, it shows that the only stationary set for the class $\mathscr{A}'$ of approximate derivatives of arbitrary approximately differentiable functions on [a, b] (infinite values allowed for f'_{ap}), or for the class $\mathscr{D}'$ of Dini derivatives of arbitrary finite functions on [a, b], or for the class $\mathscr{U}'$ of upper derivatives of arbitrary functions on [a, b], is the interval [a, b] itself.

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