LACUNARY POWER SERIES AND UNIVALENT FUNCTIONS

Ch. Pommerenke

1. M. and G. Weiss [6] have proved the existence of a number λ_0 with the following property: If $n_{k+1}/n_k > \lambda > \lambda_0$ and the function

(1) $$f(z) = a_0 + \sum_{k=1}^{\infty} a_{n_k} z^{n_k}$$

is analytic in $|z| < 1$, then either $\Sigma\, |a_n| < \infty$ or $f(z)$ assumes every value infinitely often.

J.-P. Kahane, M. Weiss, and G. Weiss [4, Theorem III'] have shown: If the coefficients a_n of the function (1) tend to 0 and $n_{k+1}/n_k > \lambda > 1$ for some $\lambda > 1$, then either $\Sigma\, |a_n| < \infty$, or the sum $f(e^{i\theta})$ (θ real) of the power series assumes every value infinitely often.

The following theorem does not yield as strong an assertion, but its hypothesis is also weaker, and its proof is shorter and quite different.

THEOREM. *Let* p *be a natural number, let*

(2) $$n_{k+1}/n_k > \lambda > 1 \qquad (k = 1, 2, \cdots),$$

and let the function

$$f(z) = a_0 + \sum_{k=1}^{\infty} \left(a_{n_k - p} z^{n_k - p} + \cdots + a_{n_k} z^{n_k}\right)$$

be analytic and different from 0 *throughout* $|z| < 1$. *Then the* $a_{n_k - q}$ $(q = 0, \cdots, p)$ *tend to* 0 *as* $k \to \infty$.

2. We shall need the following lemma, which is essentially due to Fejér and Fekete [3, p. 301] (for the case of a lacunary polynomial).

LEMMA. *Let* $\Sigma_{k=1}^{\infty}\, 1/m_k < \infty$, *and let*

$$g(z) = b_0 + \sum_{k=1}^{\infty} b_{m_k} z^{m_k}$$

be analytic and different from 0 *for* $|z| < 1$. *Then, for* $k = 1, 2, \cdots$, *the function*

(3) $$g_k(z) = b_0 + \sum_{\mu=1}^{k} \left[\prod_{j=k+1}^{\infty} \left(1 - \frac{m_\mu}{m_j}\right)\right] b_{m_\mu} z^{m_\mu}$$

is different from 0 *for* $|z| < 1$.

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Proof. Let ρ_k be the modulus of the smallest zero of $g_k(z)$. Then $1/\rho_k$ is the modulus of the largest zero of the polynomial $z^{m_k} g_k(1/z)$. Hence all zeros of its derivative lie in $\{|z| \leq 1/\rho_k\}$. Therefore the relation

$$\frac{d}{dz}[z^{m_k} g_k(1/z)] = m_k z^{m_k - m_{k-1} - 1} \cdot z^{m_{k-1}} g_{k-1}(1/z)$$

implies that $1/\rho_{k-1} \leq 1/\rho_k$ and $\rho_k \leq \rho_{k-1}$. Thus the sequence $\{\rho_k\}$ is decreasing. The definition (3) shows that $g_k(z) \to g(z)$ as $k \to \infty$, almost uniformly in $|z| < 1$ (that is, uniformly on every compact subset of $|z| < 1$). Since $g(z) \neq 0$ for $|z| < 1$, it follows that

$$\rho_k \geq \lim_{\mu \to \infty} \rho_\mu \geq 1.$$

3. *Proof of the theorem.* For $1 \leq \mu \leq k$, $1 \leq \nu \leq p$, let

$$\sigma_k(\mu, \nu) = \prod_{j=k+1}^{\infty} \prod_{q=0}^{p} \left(1 - \frac{n_\mu - \nu}{n_j - q}\right). \tag{4}$$

It follows from (2) that for $j \geq k + 1$ and sufficiently large k

$$\frac{n_\mu - \nu}{n_j - q} \leq \frac{n_k}{n_j - p} \leq \alpha^{j-k},$$

where $1/\lambda < \alpha < 1$. Hence

$$\sigma_k(\mu, \nu) \geq \prod_{j=k+1}^{\infty} (1 - \alpha^{j-k})^{p+1} = \prod_{\ell=1}^{\infty} (1 - \alpha^{\ell})^{p+1} = \sigma_0 > 0. \tag{5}$$

By the lemma, all zeros of the polynomial

$$f_k(z) = a_0 + \sum_{\mu=1}^{k} \sum_{\nu=0}^{p} \sigma_k(\mu, \nu) a_{n_\mu - \nu} z^{n_\mu - \nu} \tag{6}$$

lie in $|z| \geq 1$. Let

$$\begin{aligned} f_k^*(z) = z^{n_k} \overline{f_k(1/\bar{z})} &= \sigma_k(k, 0)\bar{a}_{n_k} + \cdots + \sigma_k(k, p)\bar{a}_{n_k - p} z^p \\ &\quad + \sigma_k(k-1, 0)\bar{a}_{n_{k-1}} z^{n_k - n_{k-1}} + \cdots. \end{aligned} \tag{7}$$

Then the function

$$h_k(z) = f_k^*(z) / f_k(z)$$

is analytic in $|z| < 1$, and $|h_k(z)| = 1$ for $|z| = 1$. Therefore it follows that $|h_k(z)| \leq 1$ for $|z| < 1$.

Suppose that a_n does not tend to 0. Then there exist sequences $\{k_m\}$ and $\{q_m\}$ $(0 \leq q_m \leq p)$ such that

$$a_{n_{k_m}-q_m} \not\to 0.$$

We may assume that $q_m \equiv q$ and that $h_{k_m}(z) \to h(z)$, almost uniformly in $|z| < 1$. It follows from (4) and (6) that $f_k(z) \to f(z)$ almost uniformly. Hence

$$f^*_{k_m}(z) = h_{k_m}(z) f_{k_m}(z)$$

also converges almost uniformly. We see that if $g(z)$ is its limit function, then

$$g(z) = h(z) f(z), \tag{8}$$

and since $n_k - n_{k-1} \to \infty$ as $k \to \infty$, it follows from (7) that $g(z)$ has the form

$$g(z) = b_0 + \cdots + b_p z^p. \tag{9}$$

The assumption and inequality (5) imply that

$$|b_q| = \lim_{m\to\infty} \sigma_{k_m}(k_m, q) |a_{n_{k_m}-q}| > 0.$$

Therefore $g(z) \not\equiv 0$, and $h(z) \not\equiv 0$ by (8). Hence

$$0 < \gamma = \sup_{|z|<1} |h(z)| \leq 1. \tag{10}$$

Let c be any complex number with $|c| < 1/\gamma$. Then

$$f(z, c) = f(z) + cg(z) = f(z)[1 + ch(z)] \neq 0 \tag{11}$$

for $|z| < 1$. By (9), the power series of $f(z)$ and $f(z, c)$ differ only in the first $p + 1$ terms. Just as for $f(z)$, we form $f^*_k(z, c)$. Then

$$f^*_k(z, c) = h_k(z, c) f_k(z, c),$$

where $h_k(z, c)$ is analytic in $|z| < 1$, and $|h_k(z, c)| \leq 1$. Let E be a denumerable dense subset of $\{c\colon |c| < 1/\gamma\}$. We may assume that $h_{k_m}(z, c) \to h(z, c)$, as $k \to \infty$, for each $c \in E$. Since only the last $p + 1$ terms of $f^*_k(z, c)$ are different from those of $f^*_k(z)$, it follows that

$$g(z) = h(z, c) f(z, c) \qquad (c \in E).$$

Hence equations (8) and (11) show that for $|z| < 1$ and $c \in E$,

$$|h(z) f(z)| = |g(z)| \leq |f(z, c)| = |[1 + ch(z)] f(z)|.$$

Since E is dense in $\{|c| \leq 1/\gamma\}$, this inequality holds for all c with $|c| \leq 1/\gamma$. It follows that if z is fixed $(|z| < 1)$ and we choose $c = \gamma^{-1} e^{i\theta}$ with suitable θ, then

$$|h(z)\,f(z)| \leq \left(1 - \frac{1}{\gamma}\,|h(z)|\right)|f(z)|.$$

Because $f(z) \neq 0$ we find, in contradiction to (10), that

$$\sup_{|z|<1} |h(z)| \leq \frac{\gamma}{1+\gamma} < \gamma.$$

This completes the proof of the theorem.

4. An analytic function is called *locally univalent* in a region if its derivative does not vanish there (every univalent function is locally univalent).

COROLLARY 1. *Let* $n_{k+1}/n_k > \lambda > 1$, *and let the function*

$$f(z) = z + \sum_{k=1}^{\infty} \left(a_{n_k-p} z^{n_k-p} + \cdots + a_{n_k} z^{n_k}\right)$$

be analytic and locally univalent in $|z| < 1$. *Then* $a_{n_k-q} = o(1/n_k)$ $(q = 0, \cdots, p;\ k \to \infty)$.

COROLLARY 2. *Let* $n_{k+1}/n_k > \lambda > 1$, *and let the function*

$$f(z) = z + \sum_{k=1}^{\infty} \left(a_{n_k-p} z^{-(n_k-p)} + \cdots + a_{n_k} z^{-n_k}\right)$$

be locally univalent and analytic in $|z| > 1$, *except for the pole at* ∞. *Then* $a_{n_k-q} = o(1/n_k)$ $(q = 0, \cdots, p;\ k \to \infty)$.

These corollaries follow at once if we apply the theorem to $f'(z)$, respectively $f'(1/z)$.

If the gap conditions are dropped, then the estimates of the corollaries need not hold. This is obvious for Corollary 1. As to Corollary 2, Clunie [1] has shown the existence of a function $f(z) = z + \cdots + a_n z^{-n} + \cdots$ univalent and analytic in $|z| > 1$ (except at ∞) for which $a_n \neq O(n^{\alpha-1})$ for some $\alpha > 0$.

5. Finally, we show that the estimate $a_n = o(1)$ of the theorem and the estimates $a_n = o(1/n)$ of the corollaries are best possible.

Let $\eta_n \downarrow 0$. Choose n_k so that $n_{k+1}/n_k \to \infty$ and $\sum_{k=1}^{\infty} \eta_{n_k} < 1$. The function

$$f(z) = 1 + \sum_{k=k_0}^{\infty} \eta_{n_k} z^{n_k}$$

does not vanish in $|z| < 1$ and satisfies the gap condition (2) for each λ, if k_0 is sufficiently large.

Also, the functions

$$f(z) = z + \sum_{k=1}^{\infty} n_k^{-1} \eta_{n_k} z^{n_k} \quad \text{and} \quad f(z) = z + \sum_{k=1}^{\infty} n_k^{-1} \eta_{n_k} z^{-n_k}$$

are starlike and therefore univalent in $|z| < 1$ and $|z| > 1$, respectively [2], [5]

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Mathematical Institute, Göttingen