TRANSITION GRAPHS AND THE STAR-HEIGHT OF REGULAR EVENTS

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1. INTRODUCTION

Kleene [3] was the first to introduce the concept of regularity for expressions and sets of words. Others, including Copi, Elgot, and Wright [2], Myhill [6], and McNaughton and Yamada [5] have discussed this topic and its relation to finite automata. (We refer the reader especially to [2] for a presentation of regularity which is similar to ours.) We show that there exist regular sets of arbitrarily large star-height. Our second main result yields as a corollary an analysis theorem for finite automata which provides an upper bound for star-height of the behavior of the automaton in terms of the "cycle complexity" of the automaton's state graph. Our first result is then used to show that in some sense this latter result is best possible.

In Section 2 we define our concepts, introduce some notation, and prove some preliminary results. In Section 3, we show that for each positive integer n, there exists a regular set of star-height n. In fact, we obtain certain sufficient conditions for a regular set to have star-height greater than or equal to n. Some related results are also obtained. In Section 4, we show that for any directed graph of cycle rank n, the set of paths between any two points is a regular set of star-height no greater than n. We also show that equality is possible. In Section 5, we give a reasonable definition of feedback for a finite automaton, and we relate it to star-height.

2. PRELIMINARIES

In this section we give some definitions, introduce some notation and conventions, and prove some preliminary results.

Let $\mathscr{A}$ be a finite set of objects, say $\mathscr{A} = \{A_1, A_2, \cdots, A_n\}$. Let $\mathscr{A}^*$ be the free semigroup with identity generated by $\mathscr{A}$, where we write the operation multiplicatively and denote it by juxtaposition. We denote by A^m the element $\underbrace{AAA \cdots AA}_{m \text{ times}}$.

Let θ denote the identity element, so $\theta A_i = A_i \theta = A_i$ for all $i = 1, \cdots, n$. We shall also call $\mathscr{A}^*$ the set of words on the alphabet $\mathscr{A}$ and hence call an element of $\mathscr{A}^*$ a word (on the alphabet $\mathscr{A}$).

Let $\mathscr{S} = \mathscr{A} \cup \{\Lambda, \Theta\}$, where Λ and Θ are distinct objects not in $\mathscr{A}$. Let "$\vee$" and "$\cdot$" be associative binary operations and "$*$" a unary operation, and define $\mathfrak{S} = (\mathscr{S}, \vee, \cdot, *)$ to be the free algebra (cf. Birkhoff [1, p. vii]) generated by $\mathscr{S}$ with operations $\vee$, $\cdot$, and $*$. Thus $\mathscr{S} \subseteq \mathfrak{S}$, and $\sigma, \omega \in \mathfrak{S}$ implies $\sigma \vee \omega$, $\sigma \cdot \omega$, and

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$\sigma^* \in \mathfrak{S}$. We call any element of $\mathfrak{S}$ a *regular expression* (on $\mathscr{S}$). As usual, we shall write $\sigma\omega$ for $\sigma \cdot \omega$.

We next define the concept of regular set (or regular event). Define the mapping $||$ from $\mathfrak{S}$ into the set of all subsets of $\mathscr{A}^*$ inductively as follows: (we read $|\sigma|$ as "the set denoted by σ")

(i) $|A_i| = \{A_i\}$ $(i = 1, \cdots, n)$, $|\Lambda| = \emptyset$, $|\Theta| = \{\theta\}$. ($\emptyset$ denotes the empty set.)

(ii) For $\sigma, \omega \in \mathfrak{S}$,

$$|\sigma \vee \omega| = |\sigma| \cup |\omega|,$$

$$|\sigma\omega| = \{xy;\ x \in |\sigma|,\ y \in |\omega|\},$$

$$|\sigma^*| = |\Theta| \cup |\sigma| \cup |\sigma\sigma| \cup |\sigma\sigma\sigma| \cup \cdots .$$

Thus $\mathscr{A}^* = |(A_1 \vee A_2 \vee \cdots \vee A_n)^*|$, which justifies our use of this notation. In fact, if $\mathscr{B}$ is any set (finite or infinite) of words, by $\mathscr{B}^*$ we shall mean the semigroup with identity generated by $\mathscr{B}$.

We define any subset of $\mathscr{A}^*$ to be an *event*. We say that an event Σ is *regular* in case there exists a regular expression $\sigma \in \mathfrak{S}$ such that $|\sigma| = \Sigma$. Thus the class of *regular events* (or *regular sets*) is the range of the function $||$ just defined. It is well known ([2, 3, 5]) that the class of regular events is a Boolean Algebra of sets. (It is interesting to note that there is no known proof of this fact which does not use the theory of finite automata. A proof using only the definition given above would be interesting.)

Finally we define the concept of star-height. We define the function $h: \mathfrak{S} \to Z$, (where Z denotes the integers) inductively as follows:

(1) For $s \in \mathscr{S}$, $h(s) = 0$

(2) For $\sigma, \omega \in \mathfrak{S}$,

$$h(\sigma\omega) = h(\sigma \vee \omega) = \max\{h(\sigma), h(\omega)\}$$

$$h(\sigma^*) = h(\sigma) + 1.$$

For example:

$$h(A_1^*(A_2 \vee A_3^*)^*) = 2,$$

$$h(A_1^* \vee (A_1^* A_2 \vee A_3)^*) = 2.$$

We call $h(\sigma)$ the *star-height* of σ. If Σ is a regular event, we define the *star-height* of Σ by

$$h(\Sigma) = \min\{h(\sigma):\ |\sigma| = \Sigma\}.$$

It is clear that for each integer n there exists a regular expression of star-height n. Our first result is that for each integer n, there exists a regular *event* of star-height n. We, in fact, show somewhat more; namely, that if a regular event Σ contains words of a certain type, then $h(\Sigma) \geq n$.

We complete this section by describing these words and proving some preliminary lemmas. Let $A_1, A_2, \cdots, A_n, \cdots$ and $B_1, B_2, \cdots, B_n, \cdots$ be infinite sequences of distinct letters. Let $\mathscr{S}_1 = \{A_1\}$, $\mathscr{T}_1 = \{A_2\}$; and for $k \geq 2$, let

$$\mathscr{S}_k = \mathscr{S}_{k-1} \cup \mathscr{T}_{k-1} \cup \{A_j\} \cup \mathscr{B}_k,$$

where $j = 2^k - 1$ and where $\mathscr{B}_k$ is some finite set of B_i's. Let

$$\mathscr{T}_k = \{A_{j+1}, \cdots, A_{2j}\} \cup \mathscr{B}'_k,$$

where $\mathscr{B}'_k$ is a finite set of B_i's such that $\mathscr{T}_k \cap \mathscr{S}_k = \emptyset$. $\mathscr{S}$ will denote a finite alphabet, but we shall presume that $\mathscr{S}_k \subseteq \mathscr{S}$ whenever we speak of $\mathscr{S}_k$. Thus, although $\mathscr{S}$ may become arbitrarily large, at any point in our discussion it will be finite. All sets and expressions considered will be presumed to be regular subsets of $\mathscr{S}^*$ or regular expressions on $\mathscr{S}$. Throughout the rest of this section and the next section small Greek letters will denote regular expressions, capital Greek letters the corresponding regular sets, lower case Roman letters will denote words, and upper case Roman letters will denote letters of the alphabet $\mathscr{S}$.

If $\Sigma \subseteq \mathscr{S}^*$, let

$$\overline{\Sigma} = \left\{a \in \mathscr{S}^*;\quad \text{for some } x, y, xay \in \Sigma\right\},$$

that is, $\overline{\Sigma}$ is the set of all subwords of words of Σ.

We define the properties Φ_k $(k = 1, 2, \cdots)$ on the class of regular sets (relative to $\mathscr{S}$) as follows: Let Σ be a regular set. Σ has property Φ_1 (relative to the letter A_1) in case for all integers n, there exists an integer $m \geq n$ such that $A_1^m \in \overline{\Sigma}$. Thus Σ has property Φ_1 on the letter A_1 if Σ contains arbitrarily long strings of the letter A_1 as subwords. A word of the form A_1^m for some $m \geq n$ is called a *1-word for exponent* n (*on the set* A_1).

We proceed by induction. Suppose that u is a (k - 1)-word for exponent n on $\mathscr{S}_{k-1}$ and w is a (k - 1)-word for exponent n on $\mathscr{T}_{k-1}$; let $j = 2^k - 1$. Then any word on the letters of $\mathscr{S}_k$, containing as a subword a word of the form $(xuywA_j)^m$, where $m \geq n$ and $x, y \in \mathscr{S}_k^*$, is a k-word for exponent n. (For example,

$$(BA_1^4 CDA_2^5 A_3)^6$$

is a 2-word for exponent 4.) We say that Σ has property Φ_k relative to $\mathscr{S}_k$ in case:

(i) Σ contains k-words for arbitrarily large exponent n on $\mathscr{S}_k$;

(ii) Σ has property Φ_{k-1} relative to each of the sets $\mathscr{S}_{k-1}$ and $\mathscr{T}_{k-1}$; and

(iii) for $T \in \mathscr{T}_{k-1}$, $S \in \mathscr{S}_{k-1}$, $TS \notin \overline{\Sigma}$.

This last condition is the crux of the construction and insures that no word beginning with letters of $\mathscr{S}_{k-1}$ follows a word ending with letters from $\mathscr{T}_{k-1}$. To say a set Ω has the property Φ_k relative to $\mathscr{T}_k$ means that the set Σ, obtained from Ω by subtracting $2^k - 1$ from the subscript of each letter A, has the property Φ_k relative to $\mathscr{S}_k$. Notice also that condition (ii) simply insures that condition (iii) holds with k - 1 replaced by m, for each $m = 1, 2, \cdots, k - 1$.

Thus, for example, Σ has property Φ_2 on $\mathscr{S}_2$ if for all n there exist $k, \ell, m \geq n$ such that $(A_1^k A_2^\ell A_3)^m \in \overline{\Sigma}$ and $A_2 A_1 \notin \overline{\Sigma}$. We remark here that it is clear that if Σ contains no k-subword for exponent m, then $\Sigma^2 = \{\sigma_1 \sigma_2;\ \sigma_1, \sigma_2 \in \Sigma\}$ contains no k-subword for exponent 2m.

We next state two lemmas, the first of which follows from the fact that concatenation distributes over union.

LEMMA 1. *If* $h(\Sigma) = n$, *then there exists a regular expression* σ *which denotes* Σ *such that*

$$\sigma = \gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_m,$$

where each γ_i *is of the form*

$$x_1 \alpha_1^* x_2 \alpha_2^* \cdots x_s \alpha_s^* x_{s+1}, \tag{1}$$

where $x_j \in \mathscr{S}^*$ *and* $0 \leq h(\alpha_j) \leq n - 1$, *for each* j.

LEMMA 2. *If* $|\gamma_1 \vee \gamma_2 \vee \cdots \vee \gamma_m|$ *has property* Φ_k, *then for some* j, $|\gamma_j|$ *has property* Φ_k. *If* $|x_1 \alpha_1^* \cdots x_s \alpha_s^* x_{s+1}|$ *has property* Φ_k, *then so does some* $|\alpha_j^*|$.

Proof. The first statement is clear. Let

$$\gamma = x_1 \alpha_1^* \cdots x_s \alpha_s^* x_{s+1}.$$

If we suppose that $|\gamma|$ has Φ_k, then clearly each $|\alpha_j^*|$ satisfies (iii) of the definition of Φ_k, and, in fact, (iii) of the definition of Φ_m for $\mathscr{S}_m$ and $\mathscr{T}_m$, for any $m = 1, 2, \cdots, k - 1$. Thus condition (ii) will be satisfied if condition (i) is satisfied. Now each of the x_i's is a word and hence of finite length. Thus, since $|\gamma|$ contains arbitrarily long subwords of a certain type, that is $|\gamma|$ contains k-subwords for arbitrarily large n, at least one of the $|\alpha_i^*|$ must also have this property. Such an $|\alpha_i^*|$ will then have property Φ_k.

DEFINITION. *Let* P_k *be the following proposition: If* Ω *is a regular event and if* Ω *satisfies* Φ_k, *then* $h(\Omega) \geq k$.

We shall show by induction in the next section that P_k is true for all k. To simplify this proof, we now prove a lemma.

LEMMA 3. *Let* $k \geq 1$ *be an integer, and suppose that proposition* P_k *is true. If* $h(\Sigma) \leq k - 1$ *and if* Σ^* *has property* Φ_k *relative to* $\mathscr{S}_k$, *then* Σ *contains a word on the letters of* $\mathscr{S}_k$.

Proof. Since P_k is true and $h(\Sigma) \leq k - 1$, Σ cannot have property Φ_k. However, Σ^* has property Φ_k so that the only way for Σ not to have property Φ_k is for there to exist an integer N such that Σ contains no k-subword for exponent N. (Note that all k-words are relative to $\mathscr{S}_k$.) Recall that by a previous remark, Σ^2 contains no k-subword for exponent 2N. Suppose now that Σ contains no word on $\mathscr{S}_k$. Then any word of Σ, and hence Σ^2, which contains the k-subword t, is of the form xty, where at least one of x and y contains a letter not in $\mathscr{S}_k$. If further, $xty \in \Sigma^2$ and t is for exponent N, then both x and y contain a letter not in $\mathscr{S}_k$. This follows, since xty must be of the form $x't_1t_2y'$, where $x't_1, t_2y' \in \Sigma$ and x' and y' contain letters not in $\mathscr{S}_k$. Thus Σ^3 also contains no k-subword for exponent 2N. By induction we obtain that, for all n, Σ^n contains no k-subword for exponent 2N so Σ^* also has this property. But this contradicts the fact that Σ^* has Φ_k.

3. REGULAR SETS OF STAR-HEIGHT n

In this section we prove the first theorem and give some examples.

THEOREM 1. P_k *is true for all positive integers* k.

Proof. The proof is by induction. If Ω satisfies Φ_1, then clearly Ω is infinite. Thus $h(\Omega) \geq 1$, and this establishes P_1.

Suppose then that P_k is true. Let Ω have property Φ_{k+1} relative to $\mathscr{S}_{k+1}$, but suppose that $h(\Omega) \leq k$. Since Ω also has property Φ_k, it follows that $h(\Omega) = k$. Let ω denote the set Ω. By Lemma 1, we may assume that $\omega = \gamma_1 \vee \cdots \vee \gamma_m$, where each γ_i has the form (1) of Lemma 1. By Lemma 2, some γ_i, say γ_1, has Φ_{k+1}. Let $\gamma_1 = x_1 \alpha_1^* \cdots x_s \alpha_s^* x_{s+1}$, where each $x_i \in \mathscr{S}^*$ and $h(\alpha_i) \leq k - 1$. By Lemma 2, some α_i^*, say α_r^*, has property Φ_{k+1}. Hence α_r^* has property Φ_k relative to $\mathscr{S}_k$ and also Φ_k relative to $\mathscr{T}_k$. Thus by Lemma 3, since $h(|\alpha_r|) \leq h(\alpha_r) \leq k - 1$, we see that $|\alpha_r|$ contains a word, say a, on the letters of $\mathscr{S}_k$, and $|\alpha_r|$ also contains a word, say b, on the letters of $\mathscr{T}_k$. Then $ba \in |\alpha_r^*|$, so there exist $T \in \mathscr{T}_k$ and $S \in \mathscr{S}_k$ such that $TS \in |\alpha_r^*| \subseteq \overline{\Omega}$. But this contradicts condition (iii) of property Φ_{k+1}. Therefore $h(\Omega) \geq k + 1$. This completes the proof of the theorem.

COROLLARY 1. *For any positive integer* k, *there exists a regular set of star-height* k.

Proof. Let $\beta_1 = A_1$, let $\beta_2 = (A_1^* A_2^* A_3)$, and let $\gamma_1 = A_2$, $\gamma_2 = (A_4^* A_5^* A_6)$. Then define β_k and γ_k inductively by

$$\beta_k = (\beta_{k-1}^* \gamma_{k-1}^* A_{2^k - 1}),$$

and γ_k is obtained from β_k by adding $2^k - 1$ to each subscript. Thus γ_k is on the letters of $\mathscr{T}_k$. The sets $|\beta_k^*|$ clearly satisfy property Φ_k and $h(\beta_k^*) = k$, so $h(|\beta_k^*|) = k$.

We would next like to note one obvious generalization of property Φ_k for which P_k is also true. We may replace the occurrence of the A_j in the definition of a k-word by a word, say b_k, on the letters of $\mathscr{S}$, provided of course that condition (iii) is not violated. Thus, for example, we could have the following 3-word for exponent 5

$$[(A_1^5 B A_2^{10} (A_2 A_3 A_1 A_3))^6 (A_4^{11} C A_5^9 A_6 A_4 A_6 A_5 A_6)^{100} (A_4 A_6 A_7 A_1 A_3)]^{10},$$

and a typical expression would be $((A_1^* B A_2^* b_2)^* (A_4^* C A_5^* c_2)^* b_3)^*$. Note that (iii) requires the words b_k and c_k to contain the letters A_j and A_{2j}, respectively, where $j = 2^k - 1$.

We also feel that it is "clear" that, by an appropriate coding, one can obtain a regular event of arbitrarily high star-height on the two letter alphabet $\{0, 1\}$. For example, carry A_n into 01^n (zero followed by n ones). (Any word from the resulting event is uniquely deconcatenable into letters of this form 01^n.) However, we have been unable to devise a proof of this conjecture. In fact, even though $|(A^* B C^* D)^*|$ has height two, it is certainly not clear that $|(0^* 10^* 1)^*|$ has height two (although R. McNaughton has informed me that he has a proof). It seems very hard to be specific about which letters in a word of an event come from which letters in an expression denoting that event.

Before proving the final corollary, we prove a lemma which may be of some independent interest. Recall that regular events form a Boolean algebra. We use $\sim$ and $\cap$ to denote relative complement and intersection, respectively.

LEMMA 4. *Let* α *and* β *be regular expressions. Then*

(2) $$|(\alpha \vee \beta)^*| \sim |\alpha\alpha^*| = |(\alpha^* \beta \alpha^*)^*|,$$

if and only if

(3) $$|(\alpha^* \beta \alpha^*)^*| \cap |\alpha \alpha^*| = \emptyset.$$

Proof. That (2) $\Rightarrow$ (3) is trivial. Suppose that (3) holds. Then clearly

$$|(\alpha^* \beta \alpha^*)^*| \subseteq |(\alpha \vee \beta)^*| \sim |\alpha \alpha^*|.$$

Let the word a be in $|(\alpha \vee \beta)^*| \sim |\alpha\alpha^*|$. Then a must be of the form $a_1 b a_2$, where $a_1, a_2 \in |(\alpha \vee \beta)^*|$ and $b \in |\beta|$. But clearly such a word $a_1 b a_2 \in |(\alpha^* \beta \alpha^*)^*|$.

LEMMA 4a. *If* α *and* β *are regular expressions, if* $\theta \notin |\alpha|$, *and if no element of* $|\beta|$ *is a subword of a word of* $|\alpha\alpha^*|$, *that is, if* $|\beta| \cap |\overline{\alpha\alpha^*}| = \emptyset$, *then* $|(\alpha \vee \beta)^*| \sim |\alpha\alpha^*| = (\alpha^* \beta \alpha^*)^*|$.

We shall call an application of a star to a regular expression σ (or regular event Σ) *non-trivial* in case for all positive integers n, $\Sigma^n \neq \Sigma^*$, where $\Sigma = |\sigma|$. We now know by Corollary 1 that there are expressions for which the application of a star is non-trivial. However, as the next corollary shows, an application of the star may be non-trivial and yet not raise the star-height of the regular set.

COROLLARY 2. *For every positive integer* k, *there exists a regular event* Σ_k, *of star-height* k *for which the application of the star is non-trivial, but such that* Σ_k^* *also has height* k.

Proof. For $k = 1$, let $\Sigma_1 = A_1^* A_2$, in which case $|\Sigma_1^*| = |(A_1 \vee A_2)^* A_2 \vee \Theta|$. For $k \geq 1$, let β_k and γ_k be as in Corollary 1, and let $j = 2^{k+1} - 1$. Define

$$\Sigma_{k+1} = |\beta_{k+1}^*| \sim |A_j^2 A_j^*|,$$

and let

$$\pi_k = \beta_k \beta_k^* A_j \vee \gamma_k \gamma_k^* A_j \vee \beta_k \beta_k^* \gamma_k \gamma_k^* A_j,$$

so that $h(\Sigma_{k+1}) \geq k + 1$, by the theorem, and $h(\pi_k) = k$. Now

$$|\beta_{k+1}| = |\beta_k^* \gamma_k^* A_j| = |A_j \vee \beta_k \beta_k^* A_j \vee \gamma_k \gamma_k^* A_j \vee \beta_k \beta_k^* \gamma_k \gamma_k^* A_j| = |A_j \vee \pi_k|$$

(simply replace β_k^* and γ_k^* by $(\Theta \vee \beta_k \beta_k^*)$ and $(\Theta \vee \gamma_k \gamma_k^*)$ and perform the multiplication). Thus, by applying Lemma 4a, we obtain the relation

$$|\beta_{k+1}^*| \sim |A_j A_j^*| = |(A_j^* \pi_k A_j^*)^*|.$$

Hence,

$$\Sigma_{k+1} = |\beta_{k+1}^*| \sim |A_j^2 A_j^*| = |(A_j^* \pi_k A_j^*)^* \vee A_j|,$$

and $h((A_j^* \pi_k A_j^*)^* \vee A_j) = k + 1$, so $h(\Sigma_{k+1}) = k + 1$. But $\Sigma_{k+1}^* = \beta_{k+1}^*$ also has star-height $k + 1$.

Note that the example Σ_1 given in this corollary has the properties:

$$\Sigma_1^j \cap \Sigma_1^i = \emptyset \quad \text{for } i \neq j,$$

and

for any regular event Ω, $h(\Omega) < h(\Sigma_1^*) \Rightarrow \Omega^* \neq \Sigma_1^*$.

The first follows simply because every word in Σ_1^j contains exactly j occurrences of the letter A_2. The second follows because Σ_1^* satisfies Φ_1 for the letters A_1 and A_2, so that if $h(\Omega) = 0$ and $\Omega^* = \Sigma_1^*$, then Ω must contain a word on the letter A_1, by Lemma 3, which is a contradiction.

The referee has kindly pointed out that for each k there are regular events Γ_k, such that

(P1) $$\Gamma_k^j \cap \Gamma_k^i = \emptyset \quad (i \neq j),$$

and

(P2) for any regular event Ω, $h(\Omega) < h(\Gamma_k^*) \Rightarrow \Omega^* \neq \Gamma_k^*$.

In particular, let α and $|\alpha|$ be of height k and such that $|\alpha^*|$ is of height k + 1. Let B be a letter not in any word of $|\alpha|$. Finally, let $\Gamma_k = |\alpha^* B|$, so

$$\Gamma_k^* = |(\alpha^* B)^*| = |(\alpha \vee B)^* B \vee \Theta|$$

has height no more than k + 1. Moreover, if Γ_k^* has height less than k + 1, say $\Gamma_k^* = |\gamma|$, where $h(\gamma) \leq k$, then the result γ' of substituting Θ for B in γ is of height no greater than k. But $|\gamma'| = |\alpha^*|$, which is a contradiction. Thus $h(\Gamma_k^*) = k + 1$. Now Γ_k satisfies (P1) since every word in Γ_k^j has exactly j occurrences of the letter B. To show (P2), we prove its contrapositive. Suppose that $\Omega^* = \Gamma_k^*$. For any ω such that $|\omega| = \Omega$ if ω' is the result of substituting Θ for B in ω, $|\omega'| = |\alpha^*|$. Consequently every such ω, and hence Ω, must be of star height no less than k + 1.

4. DIRECTED GRAPHS AND REGULAR EVENTS

In this section we relate the star-height of the set of paths between two points of a directed graph to the cycle complexity of the graph, and we give some applications. We begin with some definitions.

A *directed graph* $\mathcal{G}$ is an ordered pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$, the *points* or vertices of $\mathcal{G}$, is a set and $\mathcal{E}$, the *edges* of $\mathcal{G}$, is a subset of $\mathcal{V} \times \mathcal{V}$, the cartesian product of $\mathcal{V}$ with itself. Hence the statement $(m, n) \in \mathcal{E}$ implies $m, n \in \mathcal{V}$, and there is an edge from m to n in $\mathcal{G}$. We shall henceforth use A_{mn} to denote the edge from the point m to the point n rather than (m, n); and, in fact, capital Roman letters will generally denote edges. We call m the *initial point* and n the *terminal point* of the edge A_{mn}. We shall call $\mathcal{G}$ a finite digraph if $\mathcal{V}$ is finite.

A *path* (from m to n) is a sequence $A_1 A_2 \cdots A_k$ of edges, such that the terminal point of A_i is the initial point of A_{i+1} $(i = 1, \cdots, k - 1)$, the initial point of A_1 is m, and the terminal point of A_k is n. The points m and n are the *initial* and *terminal* points, respectively, of the path. Thus a path is simply a word of a special type on the alphabet $\mathcal{E}$.

If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and if $m \in \mathcal{V}$, let $\mathcal{E}(m)$ be the set of edges which contain m as either a terminal point or an initial point. Define

$$\mathcal{G} \sim (m) = (\mathcal{V} \sim \{m\}, \ \mathcal{E} \sim \mathcal{E}(m)),$$

so $\mathcal{G} \sim (m)$ is simply the graph obtained from $\mathcal{G}$ by eliminating m and all edges incident to m.

If $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, then $\mathcal{G}$ is a *cycle*, if, for any two points $m, n \in \mathcal{V}$, there exists a path from m to n. A cycle $\mathcal{C}$ has *rank* 1 (or is a 1-cycle), in case there exists a point m such that $\mathcal{C} \sim (m)$ does not contain a cycle, that is, if there is no point in $\mathcal{C} \sim (m)$ for which there is a path from this point to itself. By recursion, we say that a cycle $\mathcal{C}$ has *rank* k (is a k-*cycle*) if it is not a cycle of rank less than k, there exists a point m of $\mathcal{C}$ such that $\mathcal{C} \sim (m)$ contains a cycle of rank k - 1, and all subcycles of $\mathcal{C} \sim (m)$ are of rank no greater than k - 1. If a digraph $\mathcal{G}$ has no subcycles, it is of rank 0. If $\mathcal{G}$ has subcycles, then the rank of $\mathcal{G}$ is the largest rank of any subcycle of $\mathcal{G}$. (This definition of cycle rank is somewhat stronger than that originally considered and was suggested by J. Richard Büchi. I would also like to thank Dr. Büchi and Dr. Jessee B. Wright for other stimulating conversations and for interesting me in these problems.)

For $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a finite digraph, and $m, n \in \mathcal{V}$, the set of paths from m to n is a subset of $\mathcal{E}^*$. The fact that this set is regular follows from Kleene's analysis theorem [3] for finite automata. We give a new proof below which also yields an upper bound for the star-height of the regular set in terms of the cycle rank of $\mathcal{G}$.

THEOREM 2. *If* m *and* n *are points of a finite digraph* $\mathcal{G}$ *of cycle rank* k, *then there exists a regular expression of star-height* k *which denotes the set of paths from* m *to* n.

Proof. The proof is by induction on the cycle rank of the graph. If k = 0, then there are no cycles, so there are only finitely many paths from m to n. Any finite set of paths can be denoted by a regular expression of star-height zero.

Suppose that for any two points in any finite digraph of cycle rank $t \leq k$, that the set of paths from one point to the other can be denoted by a regular expression of star-height t. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite digraph of cycle rank k + 1. For z not equal to x or y, let $\alpha(x, y, z)$ be a regular expression of minimal star-height which denotes the set of paths from x to y in the graph $\mathcal{G} \sim (z)$. (We know it exists by the induction hypothesis if z is a distinguished point of $\mathcal{G}$.) Let Γ_{xy} be the set of paths from x to y in $\mathcal{G}$. (We shall use α_{xy} as a regular expression of minimal star-height which denotes this set, after we have proved that the set is regular for a particular x and y.)

First suppose that $\mathcal{G}$ is a cycle of rank k + 1. Let m be a point of $\mathcal{G}$ with the property that $\mathcal{G} \sim (m)$ is a graph of cycle rank k. Let $p_1, p_2, \cdots, p_s$ and $q_1, q_2, \cdots, q_r$ be all the points of $\mathcal{G}$ such that $A(p_i, m), A(m, q_j) \in \mathcal{E}$. (We shall use $A(m, n)$ for A_{mn} wherever necessary, and similarly for α_{xy}.) Then

$$\Gamma_{mm} = \left| \left[\bigvee_{i=1}^{s} \bigvee_{j=1}^{r} A(m, q_j)\alpha(q_j, p_i, m)A(p_i, m) \right]^* \right| .$$

Now by the induction hypothesis, each $\alpha(q_j, p_i, m)$ has star-height no greater than k so that Γ_{mm} has star-height no greater than k + 1.

Suppose now that $n \neq m$. Then

$$\Gamma_{nn} = \left| \alpha(n, n, m) \vee \left[\bigvee_{i=1}^{s} \bigvee_{j=1}^{r} \alpha(n, p_i, m)A(p_i, m)\alpha_{mm} A(m, q_j)\alpha(q_j, n, m) \right] \right|$$

$$\Gamma_{nm} = \left| \bigvee_{i=1}^{s} \alpha(n, p_i, m)A(p_i, m)\alpha_{mm} \right|$$

and

$$\Gamma_{mn} = \left| \bigvee_{j=1}^{r} \alpha_{mm} A(m, q_j)\alpha(q_j, n, m) \right|,$$

so each of these sets is regular and of star-height no greater than $k + 1$. Finally, if also $t \neq m$, then

$$\Gamma_{tn} = \left| \alpha(t, n, m) \vee \left[\bigvee_{i=1}^{s} \bigvee_{j=1}^{r} \alpha(t, p_i, m)A(p_i, m)\alpha_{mm} A(m, q_j)\alpha(q_j, n, m) \right] \right|,$$

so $h(\Gamma_{tn}) = k + 1$.

Secondly, suppose that $\mathcal{G}$ is a digraph of rank $k + 1$. Then each subcycle of $\mathcal{G}$ has rank no greater than $k + 1$. If $n \in \mathcal{V}$ and n is not in a cycle, then $\Gamma_{nn} = \emptyset$. If n is in a cycle, then Γ_{nn} is a regular set of star-height no greater than $k + 1$, by the induction hypothesis or by the first part. If $t, n \in \mathcal{V}$, $t \neq n$, and they are in the same cycle, then Γ_{tn} is again given by the first part.

Hence we may assume that t and n are not in the same cycle. Let $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_r$ be the set of cycles between t and n. Then any path from t to n is of the form

$$xw_1(p_1, q_1)w_2 \cdots w_u(p_u, q_u)w_{u+1} y,$$

where x (respectively y) denotes a path through the cycle containing t (respectively n) with initial point t (respectively terminal point n) and may be θ if t (respectively n) is not in a cycle; w_1 denotes a path from the cycle containing t (or from t itself) to the i_1-th cycle, and w_{u+1} denotes a path from the i_u-th cycle to the cycle containing n (or to n itself); w_j denotes a path from the i_{j-1}-th cycle to the i_j-th cycle; and (p_j, q_j) denotes a path through the i_j-th cycle. Thus any path is an element of a set denoted by an expression of the form

$$(*) \qquad \beta w_1 \alpha(p_1, q_1)w_2 \cdots w_u \alpha(p_u, q_u)w_{u+1} \gamma,$$

where β (respectively γ) may be θ if t (respectively n) is not in a cycle, or else $\beta = \alpha_{tq}$ (respectively $\gamma = \alpha_{pn}$) (for some p and q). Now there can only be a finite number of paths between any two cycles so that Γ_{tn} is a finite union of sets of the form (*) and hence is of star-height no greater than $k + 1$. This completes the proof of the theorem.

Our first application of this theorem yields a somewhat strengthened analysis theorem.

COROLLARY 1. (Analysis theorem for finite automata.) *Let* $\mathfrak{M}$ *be a finite automaton with state graph,* $\mathcal{G}$. *If* $\mathcal{G}$ *has cycle rank* n, *then there exists a regular expression of star-height no greater than* n *which denotes the behavior of* $\mathfrak{M}$.

Proof. The behavior of $\mathfrak{M}$ is the set of paths from the initial point s_0 of the finite digraph $\mathcal{G}$ to any of the finite set F of terminal points. Thus by Theorem 2, the behavior of $\mathfrak{M}$ is a finite union of regular events of star-height no greater than n.

Our second corollary is related to the theory of commutative machines (cf. [4] for the definitions and known results in this area). If Ω is a set of words on the two letter alphabet $\{A, B\}$, let $c(\Omega)$ be the set of all words which can be obtained from words of Ω by permutting the letters of the word. For example,

$$c(\{AAB\}) = \{AAB, ABA, BAA\}, \quad c(|A^* B|) = |A^* BA^*|,$$

$$c(|A^* B^*|) = |(A \vee B)^*|.$$

If α is a regular expression, we shall also write $c(\alpha)$ for $c(|\alpha|)$.

COROLLARY 2. *Let* k *and* n *be non-negative integers. Then*

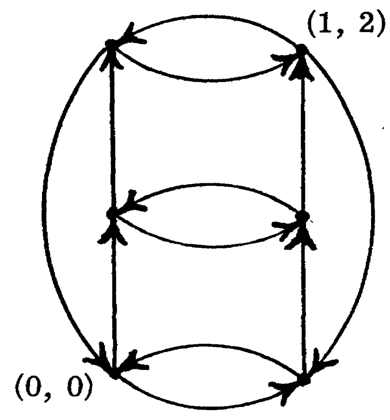
$$h(c((A^k)^* (B^n)^*)) \leq 1 + \min \{n, k\}.$$

Moreover, $h(c((A^k)^* (B^k)^*)) \leq k$.

Proof. Since $c((A^k)^* (B^n)^*) = c((B^n)^* (A^k)^*)$, we may assume without loss of generality that $k \leq n$. Consider the following set of lattice points in the plane:

$$\mathcal{V} = \{(x, y); \quad 0 \leq x \leq k - 1, \quad 0 \leq y \leq n - 1\}.$$

Construct a digraph $\mathcal{G}$ on this set of points by drawing an edge from each (x, y) to $(x, y + 1)$ and $(x + 1, y)$, provided that $(x, y + 1) \in \mathcal{V}$ and $(x + 1, y) \in \mathcal{V}$. By drawing an edge from $(k - 1, y)$ to $(0, y)$ for each $y = 0, 1, \cdots, n - 1$ and by drawing an edge from $(x, n - 1)$ to $(x, 0)$ for each $x = 0, \cdots, k - 1$. For $k = 2$ and $n = 3$, this becomes



Now label each horizontal edge with an A and each vertical edge with a B. Then it is easy to see (see [4; p. 24]) that $c((A^k)^*(B^n)^*)$ is precisely the set of all paths from $(0, 0)$ to itself.

Now if we delete the points $(0, n - 1)$, $(1, n - 1) \cdots$, $(k - 1, n - 1)$ from the digraph, we obtain a digraph of cycle rank 1 (it is a series of rank 1 cycles placed on top of each other). Thus the cycle rank of $\mathscr{G}$ is no greater than k + 1. An application of the theorem then gives the result.

If n = k, then by deleting the points (0, 0)(1, 1), ⋯, (k - 1, k - 1), we obtain a digraph of rank 0. Hence Theorem 2 again yields the result.

As our final result we show that in some sense Theorem 2 is a best possible result. We exhibit a regular event of star-height n which is the set of paths from a point to itself in a digraph (in fact a cycle) of cycle rank n.

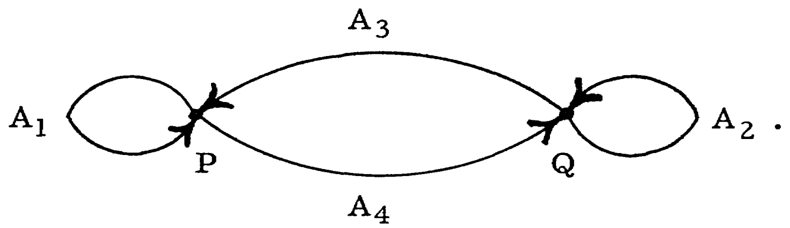
THEOREM 3. *Given a positive integer* k, *there exists a finite digraph of cycle rank* k *and a point in that graph such that the set of paths from that point to itself has star-height* k.

Proof. Let $R_k = \{A_j \mid j = 1, 2, \cdots, (3\cdot 2^{k-1} - 2)\}$. Let $\sigma_1 = A_1^*$, $\tau_1 = A_2^*$, $\sigma_2 = (A_1^* A_3 A_2^* A_4)^* A_1^*$, and $\tau_2 = (A_5^* A_7 A_6^* A_8)^* A_5^*$. By recursion, let

$$\sigma_{k+1} = (\sigma_k A_{j-1} \tau_k A_j)^* \sigma_k ,$$

where $j = 3\cdot 2^k - 2$ and τ_k is obtained from σ_k by adding $3\cdot 2^{k-1} - 2$ to the subscript of each letter appearing in σ_k. By Theorem 1, $h(|\sigma_k|) = k$, since σ_k satisfies Φ_k (on a proper subset of R_k).

To see that σ_k denotes a set of paths as claimed, first note that σ_2 denotes the set of all paths from a point to itself in a complete digraph with two points. To be specific, let P be the point such that A_1 is the edge from P to itself. Then σ_2 denotes the set of paths from P to P in the following digraph:



Now σ_3 is built as follows: Take a copy of the above digraph with edges labeled by letters obtained by adding 4 to each of the given letters and with distinguished point P' corresponding to P. Then τ_2 denotes the set of paths from P' to itself. Draw an edge from P to P', and label it A_9. Then draw an edge from P' to P, and label it A_{10}. The resulting digraph is such that the set of paths from P to P is now $|\sigma_3|$. By deleting the points P and P', we obtain a digraph of rank 1, so the original graph has rank 3. Similarly, from the graph for σ_k and τ_k (the latter being isomorphic to the former), by adding two edges (labeled with A_{j-1} and A_j, where $j = 3\cdot 2^k - 2$), we obtain a digraph with the property that the set of paths from P to P is denoted by σ_{k+1}. Again, by deleting the point P, we obtain a disconnected graph of rank k (and deleting any other point gives a graph of rank at least k) so that the rank of this graph is k + 1, as desired.

Note that we have proved that the set of paths from a point to itself in the complete digraph on two points has star-height two. Although one can easily construct a regular expression of star-height n which denotes the set of paths from a point to itself in the complete digraph on n points, we have been unable to prove (for $n \geq 3$) the eminently reasonable statement that this set actually has star-height n.

5. FEEDBACK NUMBER

There seems to be some interest in attempting to describe the complexity of a finite automaton, or its behavior, by means of its cycle structure. For example, some notion of feedback seems necessary in considering logical nets (cf. [2]). We would like to make a definition of feedback number and indicate the proof of the theorem which we feel makes this definition reasonable. This definition was suggested to us by J. Richard Büchi.

DEFINITION. *If* Σ *is a regular event, let the rank of* Σ, $\mathrm{rk}(\Sigma)$, *be the minimum of the set of cycle ranks (as defined in Section 4) of finite digraphs* $\mathcal{G}$ *with the property that there exists a point* s_0 *in* $\mathcal{G}$ *and a set* F *of points of* $\mathcal{G}$ *such that, with the appropriate labeling (allowing the empty word* θ), Σ *is the set of paths from* s_0 *to the points of* F. *If* $\mathfrak{M}$ *is a finite automaton with behavior* Σ, *let the* feedback number *of* $\mathfrak{M}$, $\mathrm{FN}(\mathfrak{M})$, *equal* $\mathrm{rk}(\Sigma)$.

We then find the following theorem.

THEOREM 4. *If* $\mathfrak{M}$ *is a finite automaton with behavior* Σ, *then* $\mathrm{FN}(\mathfrak{M})$ *is equal to the star-height of* Σ.

This theorem shows that the feedback gives a good measure of the complexity of its behavior (presuming star-height is a good measure of complexity).

Proof. (Sketch) We must show that $\mathrm{rk}(\Sigma) = \mathrm{h}(\Sigma)$. Since there is a digraph of cycle complexity $\mathrm{rk}(\Sigma)$ which realizes Σ, by Theorem 2 we find that $\mathrm{rk}(\Sigma) \geq \mathrm{h}(\Sigma)$.

To show that $\mathrm{rk}(\Sigma) \leq \mathrm{h}(\Sigma)$, let σ be a regular expression of minimal star-height which denotes Σ. It is easy to see that by taking the canonical way of building a digraph from a regular expression, using the null word θ, the only time the cycle rank can be increased is when one encounters a star. Moreover, this can be accomplished in such a way that the cycle rank is increased by at most one. This shows that $\mathrm{rk}(\Sigma) \leq \mathrm{h}(\Sigma)$ as desired.

6. CONCLUSION

We have shown the existence of regular events of arbitrarily large star-height; in fact, we have given certain sufficient conditions which will insure that an event has height greater than n. If the event is represented by means of a digraph, we have shown how to put an upper bound on its star-height.

However, we have clearly left open the significant unsolved problem of determining the star height of a given regular event as presented by a given regular expression or digraph or by some other means. What would be most desirable is some sort of "canonical form" regular expression for denoting regular events to which any regular expression could mechanically be reduced. This canonical form would give the star height of the event and also, hopefully, other information about the structure of the event. However, any means of determining the star height of the event from any expression or digraph denoting it would be a worthwhile innovation.

Another interesting question, which is related to the one raised at the end of Section 4, is the following extension of Theorem 2. For a digraph of cycle rank k, do there necessarily exist points m and n such that the set of paths from m to n is an *event* of star height k?

Finally, we would again like to draw the reader's attention to the problem (stated in the paragraph preceeding Lemma 4 in Section 3) of showing that there are events of arbitrarily large star-height on the two-letter alphabet.

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