MAPPING CYLINDER NEIGHBORHOODS

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Let (X, A) be a pair of spaces having two structures each of which induces, in some way, a neighborhood of A which is a mapping cylinder. We shall show in this paper that the two neighborhoods are *homeomorphic*. For example, let S be a differential structure on X which induces on A the structure of a differential submanifold. Then any open tubular neighborhood of A (that is, a realization of the normal bundle of A for some complete Riemannian metric on X by normal disks of sufficiently small radius) is a mapping cylinder neighborhood. There are many examples of pairs (X, A) admitting more than one such differential structure. Alternatively, if A is a full subcomplex of some triangulation T of X, then an open simplical (that is, regular) neighborhood of A in the first barycentric subdivision of T is a mapping cylinder neighborhood.

We recall that the *mapping cylinder* M_f of a map f of a space X onto a space Y is the disjoint union $X \times [0, 1] \cup Y$ with each (x, 1) identified to $f(x) \in Y$. By identifying each $x \in X$ with $(x, 0) \in M_f$, we consider X, Y as closed subsets of M_f. For any set A in a space, b(A), i(A), and Cl A will denote its set-theoretical boundary, interior, and closure, respectively. Let A be a closed subset of a space X. An open set $U \supset A$ of X is called an *open mapping cylinder neighborhood* (MCN) of A if there exists a map f of b(U) onto b(A) and a homeomorphism h of (Cl U) - i(A) onto M_f such that $h| b(U) \cup b(A) = 1$. Our main result can be stated in the following form.

THEOREM 1. *Let* U, V *be* MCN's *for a closed subset* A *of a space* X. *If* b(U) *and* b(V) *are paracompact and locally compact, then there exists a homeomorphism of* V *onto* U *that leaves pointwise fixed a neighborhood of* A.

In particular, we obtain the following corollary.

COROLLARY 1. *Let* U, V *be* MCN's *for a (not necessarily compact) closed subset* A *of a locally compact metric space* X. *Then there exists a homeomorphism of* U *onto* V *that leaves pointwise fixed a neighborhood of* A.

If A is any subcomplex of a locally finite complex X, then by the *open regular neighborhood* of A, we shall mean the simplicial neighborhood of A in the second barycentric subdivision. Here we use the term complex both for the complex itself and for the underlying topological space.

COROLLARY 2. *Let* T_1, T_2 *be two locally finite triangulations of a closed pair* (X, A). *Let* R_i *denote the open regular neighborhoods of* A *under* T_i. *Then there exists a homeomorphism of* R_1 *onto* R_2 *that leaves pointwise fixed a neighborhood of* A.

It is known [2] that the tangent spaces of a manifold M corresponding to two differentiable structures may not be equivalent as bundles over M. However, there is the following result in which M is considered as embedded in the tangent space as the zero cross-section.

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COROLLARY 3. *Let* X_1, X_2 *be tangent spaces of a manifold* M *corresponding to two differentiable structures on* M. *Then there exists a homeomorphism* h *of* X_1 *onto* X_2 *such that* $h | M = 1$.

Proofs. Theorem 2 (see the next section) is a special case of Theorem 1. Conversely, Theorem 1 is obtained from Theorem 2 as follows. Shrink A to a point x_0 and apply Theorem 2 to the resulting space. Since the homeomorphism stated in Theorem 2 does leave pointwise fixed a neighborhood of x_0, we obtain Theorem 1.

Corollaries 1 and 2 are immediate consequences of Theorem 1. Corollary 3 follows from Theorem 1 and the observation that M can be naturally identified to the diagonal $\triangle$ of $M \times M$ and any tangent space of M can be considered as an open tubular neighborhood of $\triangle$ in $M \times M$ and hence a MCN of $\triangle$ in $M \times M$.

OPEN CONE NEIGHBORHOODS

The cone C(A) over a space A is $A \times [0, 4]$ with $A \times 0$ identified to a point v which we call the vertex of the cone. In connection with cones or open cones (see below), v will stand for the vertex. The definition of the open cone OC(A) over A is entirely similar except that [0, 4) replaces [0, 4]. We identify OC(A) as the open subset of C(A) which leaves out the base $A = A \times 4$. Precisely speaking, there exists the natural projection $\eta: A \times [0, 4] \to C(A)$. But since $A \times [0, 4]$ never appears in our discussion, we suppress η and use (a, t), $A \times t$, $A \times [t_1, t_2]$ while actually meaning $\eta(a, t)$, $\eta(A \times t)$, and $\eta(A \times [t_1, t_2])$, respectively.

Let x be a point of a space X. By an *open cone neighborhood* of x, we mean any open subset U of X for which there exists a space A and a homeomorphism h of C(A) into X such that $h(v) = x$, $h(OC(A)) = U$ and $h(C(A)) = \mathrm{Cl}\, U$. Notice that this definition is stronger than the one given in [1]. However, in case X is locally compact, as was the case in [1], then the two definitions are essentially equivalent.

By $U = (A, h)$ we mean that U is an open cone neighborhood and there is a homeomorphism $h: C(A) \to \mathrm{Cl}\ U$ as in the definition.

THEOREM 2. *Let* $U = (A, h)$ *and* $V = (B, k)$ *be two open cone neighborhoods of a point* x *in a space* X. *Suppose* A *and* B *are paracompact and locally compact. Then there exists a homeomorphism of* U *onto* V *which leaves a neighborhood of* x *pointwise fixed.*

Proof. Theorem 2 is a generalization of Theorem 1 of [1]. In order to use the method of proof of [1], considerable care must be taken due to the non-compactness of A and B. Therefore, we show how to place the present situation in a setting so that the method of [1] can be used.

Let A^t (or B^t) denote the subset of C(A) (or C(B)) consisting of the points (a, t') (or (b, t')) with $t' \leq t$. We first note that if $k(B^{t_1}) \subset h(A^{t_2}) - h(A \times t_2)$, then $k(B \times t_1)$ separates $h(A \times t_2)$ from x. In fact,

$$X - k(B \times t_1) = C \cup D, \quad \text{where}$$

$$C = k(B^{t_1} - B \times t_1)$$

$$D = X - k(B^{t_1}).$$

As $k(B^{t_1})$ is a closed subset of Cl U and $k(B^{t_1} - B \times t_1)$ is an open subset of U, C and D are open and disjoint. Furthermore, $x \in C$ and $A \times t_2 \subset D$.

We divide the proof into two cases.

Case 1. Suppose the homeomorphisms h and k satisfy the following separation property.

THE SEPARATION PROPERTY. *There exist positive numbers* $p < q < r$, $s < t$ *and* ε *such that* (see Figure 1)

(1) $h(A \times p)$, $k(B \times (s - \varepsilon))$, $k(B \times s)$, $h(A \times (q - \varepsilon))$

$h(A \times q)$, $h(A \times (q + \varepsilon))$, $k(B \times t)$, $k(B \times (t + \varepsilon))$

and $h(A \times r)$ *are disjoint, and*

(2) $h(A^p) \subset k(B^{s-\varepsilon}) \subset k(B^s) \subset h(A^{q-\varepsilon}) \subset h(A^q)$

$\subset h(A^{q+\varepsilon}) \subset k(B^t) \subset k(B^{t+\varepsilon}) \subset h(A^r)$.

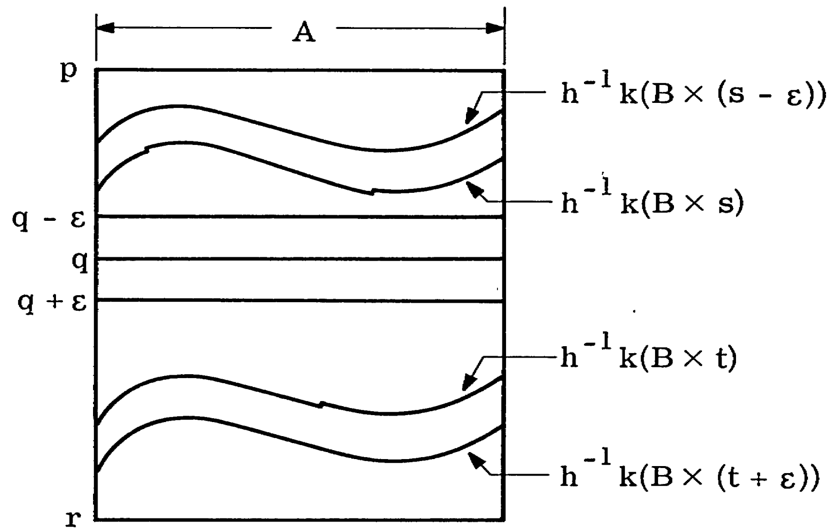

Figure 1

Let g_1 be a homeomorphism of Cl U onto itself such that

$$g_1 | h(A^{q-\varepsilon}) \cup (\mathrm{Cl}\ U - h(A^{(r+4)/2})) = 1,$$

$$g_1(h(A^q)) = h(A^r) \text{ and } g_1 h(a, q) = h(a, r) \quad (a \in A).$$

Let g_2 be a homeomorphism of Cl U onto itself such that

$$g_2 | k(B^{s-\varepsilon}) \cup (\mathrm{Cl}\ U - k(B^{t+\varepsilon})) = 1,$$

$$g_2(k(B^s)) = k(B^t) \text{ and } g_2 k(b, s) = k(b, t) \quad (b \in B).$$

Let g_3 be a homeomorphism of Cl U onto itself such that

$$g_3 | h(A^{p/2}) \cup h(A^{q+\varepsilon}) = 1,$$

$$g_3(h(A^p)) = h(A^q) \text{ and } g_3(h(a, p)) = h(a, q) \quad (a \in A).$$

Then $g = g_3 g_2 g_1 | h(A \times [p, q])$ is a homeomorphism of $h(A \times [p, q])$ onto $h(A \times [q, r])$ such that

$$g(h(a, p)) = h(a, q), \; g(h(a, q)) = h(a, r)$$

and

$$g(k(b, s)) = k(b, t).$$

As in the proof of Theorem 1 of [1], the existence of such g is enough to guarantee the conclusion of the theorem for Case 1.

Case 2. Suppose h and k do not satisfy the separation property. We will modify h and k so that they will satisfy the separation property.

Let $\{W_\alpha\}$ be a locally finite open covering of B such that each set $\mathrm{Cl}\, W_\alpha$ is compact. Such a $\{W_\alpha\}$ exists as B is paracompact and locally compact. For each W_α, there exists a positive number $e(\alpha) < 3$ such that

$$k(W_\alpha \times [0, e(\alpha)]) \subset h(A^3 - A \times 3).$$

Also there exists a family $\{f_\alpha\}$ of continuous maps $f_\alpha\colon B \to [0, 1]$ such that each f_α is zero outside W_α and $\Sigma_\alpha f_\alpha(b) = 1$ for each $b \in B$.

Define $F\colon B \to [0, 3)$ by

$$F(b) = \max_\alpha \{e(\alpha) f_\alpha(b)\}.$$

Clearly $F(b)$ is positive-valued. Furthermore, it is continuous. To see this we note that for each $b \in B$, there exists an open neighborhood U of b such that U meets at most a finite number of W's, say $W_{\alpha_1}, \cdots, W_{\alpha_k}$. Then

$$F|U = \max\{e(\alpha_1) f_{\alpha_1}, e(\alpha_2) f_{\alpha_2}, \cdots, e(\alpha_k) f_{\alpha_k}\}.$$

Observe that for each $b \in B$,

$$k(b \times [0, F(b)]) \subset k(A^3 - A \times 3).$$

Define a homeomorphism k_1 of $C(B)$ onto $\mathrm{Cl}\, V$ by

$$k_1(b, w) = \begin{cases} k(b, w) & \text{if } 3 \le w \le 4, \\ k(b, F(b) + (2w - 5)(3 - F(b)) & \text{if } 2.5 \le w \le 3, \\ k(b, 2wF(b)/5) & \text{if } 0 \le w \le 2.5. \end{cases}$$

Then $k_1 = k$ on $B \times [3, 4]$ and

$$k_1(B \times [0, t + \varepsilon]) = k_1(B \times [0, 2 + 0.5]) \subset h(A^3 - A \times 3).$$

In exactly the same way, we modify h to h_1 such that $h_1 = h$ on $A \times [3, 4]$ and $A \times [0, q + \varepsilon] = A \times [0, 2.5]$ is mapped into $k_1(B^2 - B \times 2)$. This process is iterated until the desired modifications h_i and k_j are attained.

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