UNIFORM DISTRIBUTION RELATIVE TO A FIXED SEQUENCE

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1. INTRODUCTION

In the usual theory of the distribution modulo 1 of an increasing sequence $s_1, s_2, \cdots$ of real numbers, one considers the positions of the successive terms of the sequence in the unit intervals $(n, n + 1)$ into which they fall. These positions are specified by the fractional parts of $s_1, s_2, \cdots$. The definition of uniform distribution modulo 1, in terms of these fractional parts, is well known. In an earlier paper [1], one of us considered a generalization of this concept, in which the unit intervals are replaced by the intervals (z_n, z_{n+1}) between the successive numbers of a fixed sequence $0 < z_1 < z_2 < \cdots$, where $z_n \to \infty$ with n. The fractional part of a positive real number t, relative to the sequence $\Delta = \{z_n\}$, is defined by

$$(1) \qquad \langle t \rangle_\Delta = \frac{t - z_{n-1}}{z_n - z_{n-1}} \quad \text{for } z_{n-1} \leq t < z_n .$$

A sequence $s_1, s_2, \cdots$ is said to be *uniformly distributed modulo* Δ if the proportion of $s_1, \cdots, s_N$ for which $\langle s_k \rangle_\Delta < \alpha$ has the limit α as $N \to \infty$, for each α such that $0 < \alpha < 1$.

It is reasonable to impose some condition on the sequence Δ, and we shall suppose that $z_n - z_{n-1}$ is either monotonic increasing or monotonic decreasing. In the increasing case, it was proved in [1] that the sequence $s_k = kx$ is uniformly distributed modulo Δ for each $x > 0$ provided that $z_n/z_{n-1} \to 1$ as $n \to \infty$, and that this supplementary condition is necessary. The decreasing case is more difficult; it was proved that the sequence $s_k = kx$ is uniformly distributed modulo Δ for almost all $x > 0$ (in the sense of Lebesgue measure) provided that $z_n - z_{n-1} = O(z_n^{-1})$. But this was a severe restriction on the z_n.

The main object of the present note is to prove this "almost all" result in the decreasing case without imposing any additional condition on the z_n. Although the case $s_k = kx$ is the one we have principally in mind, the method yields a more general result with little extra effort. We prove the following result. [The words "increasing" and "decreasing" are used in the wide sense henceforth.]

THEOREM. *Suppose that* $z_n - z_{n-1}$ *decreases as* n *increases, and that* $z_n \to \infty$. *Let* $a_1, a_2, \cdots$ *be any sequence of positive real numbers such that*

$$(2) \qquad a_{k+1} - a_k \geq C a_k/k \quad (C > 0).$$

Then the sequence $s_k = a_k x$ *is uniformly distributed modulo* $\Delta = \{z_n\}$ *for almost all* $x > 0$. *In particular, this holds for* $s_k = kx$ *or, more generally, for* $s_k = k^\gamma x$ *for any fixed* $\gamma > 0$.

We may remark that the condition (2) is also satisfied if $a_{k+1} - a_k$ increases with k.

The proof of the theorem makes use of the condition, given in the preceding note, for a sequence $s_k(x)$ to be uniformly distributed (mod 1) for almost all x in an interval (α, β).

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2. LEMMA

Let $\psi(x)$ *be a real function, defined for* $x > 0$ *and satisfying the conditions*

$$\psi'(x) > 0, \quad \psi''(x) \geq 0.$$

Suppose that

$$\beta > \alpha > 0, \quad p > q > 0, \quad m > 0.$$

Then

$$\left| \int_\alpha^\beta e(m\psi(px) - m\psi(qx))dx \right| \leq \frac{p}{\pi m(p - q)^2 \psi'(q\alpha)},$$

where $e(\theta)$ *denotes* $e^{2\pi i\theta}$.

Proof. Write

$$\Psi(x) = \psi(px) - \psi(qx).$$

Denoting the integral in question by J, we see that

$$2\pi imJ = \int_\alpha^\beta \frac{d\,e(m\Psi(x))}{\Psi'(x)}$$

$$= \frac{e(m\Psi(\beta))}{\Psi'(\beta)} - \frac{e(m\Psi(\alpha))}{\Psi'(\alpha)} - \int_\alpha^\beta e(m\Psi(x))d\left(\frac{1}{\Psi'(x)}\right).$$

Thus

$$2\pi m|J| \leq \frac{1}{\Psi'(\beta)} + \frac{1}{\Psi'(\alpha)} + \int_\alpha^\beta \left|\frac{d}{dx}\frac{1}{\Psi'(x)}\right| dx.$$

We have assumed tacitly that $\Psi'(x) > 0$; in fact

$$\Psi'(x) = p\psi'(px) - q\psi'(qx) \geq (p - q)\psi'(qx) > 0.$$

Putting $F(x) = 1/\Psi'(x)$ for brevity, we may write

$$F(\beta) - F(\alpha) = \int_\alpha^\beta F'(x)dx,$$

whence

$$2\pi m|J| \leq 2F(\alpha) + \int_\alpha^\beta \{F'(x) + |F'(x)|\}\, dx.$$

Now

$$F'(x) = -\frac{\Psi''(x)}{(\Psi'(x))^2} = -\frac{p^2\psi''(px) - q^2\psi''(qx)}{(p\psi'(px) - q\psi'(qx))^2} \le \frac{q^2\psi''(qx)}{(p-q)^2(\psi'(qx))^2}.$$

Hence

$$F'(x) + |F'(x)| \le \frac{2q^2\psi''(qx)}{(p-q)^2(\psi'(qx))^2};$$

for the left hand side is 0 if $F'(x) < 0$. Hence

$$2\pi m|J| \le \frac{2}{\Psi'(\alpha)} + \frac{2q^2}{(p-q)^2}\int_\alpha^\beta \frac{\psi''(qx)}{(\psi'(qx))^2}\,dx$$

$$\le \frac{2}{(p-q)\psi'(q\alpha)} + \frac{2q}{(p-q)^2\psi'(q\alpha)},$$

whence the result follows.

3. PROOF OF THE THEOREM

Let the function $\phi(t)$ be defined by

$$\phi(t) = n + \frac{t - z_{n-1}}{z_n - z_{n-1}} \quad \text{for } z_{n-1} \le t \le z_n. \tag{3}$$

Then the fractional part of $\phi(t)$ in the ordinary sense is the same as the fractional part of $t \pmod \Delta$, and therefore a sequence s_k is uniformly distributed $(\text{mod } \Delta)$ if and only if the sequence $\phi(s_k)$ is uniformly distributed (mod 1); see [1; Section 1].

Let

$$S(N, x) = \frac{1}{N}\sum_{k=1}^{N} e(m\phi(a_k x)),$$

where m is a positive integer, and let

$$I(N) = \int_\alpha^\beta |S(N, x)|^2\,dx,$$

where $\beta > \alpha > 0$. It follows from the result of the preceding note that the sequence $a_k x$ is uniformly distributed $(\text{mod } \Delta)$ for almost all x in (α, β) provided that

$$\sum_{N=1}^{\infty} \frac{1}{N} I(N) \text{ converges.} \tag{4}$$

This is to hold for each integer $m > 0$.

Now

(5) $$I(N) = \frac{\beta - \alpha}{N} + \frac{1}{N^2} \sum_{k<j\leq N} 2\,\Re\, J_{j,k},$$

where

(6) $$J_{j,k} = \int_\alpha^\beta e(m\phi(a_j x) - m\phi(a_k x))dx.$$

The function $\phi(t)$, defined in (3), is continuous, and is linear in each of the intervals $z_{n-1} \leq t \leq z_n$, its derivative in this interval being $(z_n - z_{n-1})^{-1}$. We write

$$\delta(t) = z_n - z_{n-1} \quad \text{for } z_{n-1} < t < z_n,$$

and recall that $\delta(t)$ decreases as t increases. Thus $\phi'(t) = 1/\delta(t)$ increases, except that it is undefined at the isolated points $t = z_n$.

We can obviously approximate $\phi(t)$ arbitrarily closely by a twice differentiable function $\psi(t)$ which satisfies the inequality $\psi''(t) \geq 0$, and we can also make $\psi'(t)$ approximate $\phi'(t)$ arbitrarily closely, except in arbitrarily small intervals around the points z_n. Hence the result of the lemma is applicable to the integral $J_{j,k}$ in (6) with $p = a_j$ and $q = a_k$. Thus

$$|J_{j,k}| \leq \frac{a_j}{\pi m(a_j - a_k)^2 \phi'(a_k\alpha)} = \frac{a_j\,\delta(a_k\alpha)}{\pi m(a_j - a_k)^2}.$$

(We have tacitly supposed that $a_k\alpha$ is not one of the points z_n, but there is plainly no loss of generality in this.) We also have the trivial estimate $|J_{j,k}| \leq \beta - \alpha$.

Returning to (5) and (6), we see that in order to prove the convergence of the series (4), it suffices to prove the convergence of

$$\sum_{N=1}^\infty \frac{1}{N^3} \sum_{k<j\leq N} \min\left(\frac{a_j}{(a_j - a_k)^2}, 1\right).$$

Changing the order of summation, we deduce that it suffices to prove the convergence of

$$S = \sum_{k<j} \frac{1}{j^2} \min\left(\frac{a_j}{(a_j - a_k)^2}, 1\right).$$

For this we must use the hypothesis (2). If $j = k + \ell$, we see that

(7) $$\begin{aligned} a_j - a_k &= (a_{k+1} - a_k) + \cdots + (a_{k+\ell} - a_{k+\ell-1}) \\ &\geq Ca_k(k^{-1} + \cdots + (k + \ell - 1)^{-1}) \\ &\geq C\ell a_k(k + \ell)^{-1}. \end{aligned}$$

Hence

$$a_j \geq \left(\frac{C\ell}{k + \ell} + 1\right) a_k,$$

and, since the function $x/(x - \alpha)^2$ decreases for $x > \alpha$, this implies that

$$\frac{a_j}{(a_j - a_k)^2} \leq \frac{(C\ell + k + \ell)(k + \ell)}{C^2 \ell^2 a_k}.$$

$$\ll \frac{(k + \ell)^2}{\ell^2 a_k}.$$

Hence

$$S \ll \sum_k \sum_\ell \frac{1}{(k + \ell)^2} \min\left(\frac{(k + \ell)^2}{\ell^2 a_k}, 1\right)$$

$$\ll \sum_k \sum_\ell \frac{1}{(k + \ell)^2} \frac{(k + \ell)}{\ell a_k^{1/2}}$$

$$\ll \sum_k \frac{1}{a_k^{1/2}} \sum_\ell \frac{1}{\ell(\ell + k)}$$

$$\ll \sum_k \frac{\log k}{k a_k^{1/2}}.$$

Now the hypothesis (2) implies that $a_k \gg k^\delta$ for some $\delta > 0$; for we know that

$$a_{2k} - a_k \gg a_k$$

by (7), whence $a_{2^\nu} > a_1(1 + \delta_1)^\nu$, which gives the result. Hence the series is majorized by

$$\sum_k \frac{\log k}{k^{1+\delta/2}},$$

and so converges. This completes the proof.

REFERENCE

1. W. J. LeVeque, *On uniform distribution modulo a subdivision,* Pacific J. Math. 3 (1953), 757-771.

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