THE INTERSECTION OF FIBONACCI SEQUENCES

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For the construction of a groupoid satisfying the identity $a((a \cdot ba)a) = b$ but not the identity $(a(ab \cdot a))a = b$, it was necessary to examine the intersection of Fibonacci sequences. In particular, Theorem 1 below was used. We shall prove that two Fibonacci sequences generally do not meet and that if they do meet at least three times, then one is simply the tail of the other.

If a_1 and a_2 are positive integers $(a_1 \leq a_2)$, let $F(a_1, a_2)$, or simply F, denote the Fibonacci sequence $a_1, a_2, a_3, \cdots$ whose first two terms are a_1 and a_2 (that is, $a_k = a_{k-1} + a_{k-2}$ for $k \geq 3$). Let $\overline{F}(a_1, a_2)$, or simply $\overline{F}$, be the set $\{a_1, a_2, a_3, \cdots\}$. We denote the kth term of the standard Fibonacci sequence $F(1, 1)$ by f_k. For convenience we define f_0 to be 0. A predicate P about positive integers holds "for almost all n" if $\{n: P(n) \text{ is true}\}$ has density 1, that is, if $\lim_{n \to \infty} A(n)/n = 1$, where $A(n)$ is the number of integers not exceeding n for which P is true.

THEOREM 1. *If* n *is a positive integer and* $F_1, F_2, \cdots, F_s$ *are Fibonacci sequences, then there is an integer* $m > n$ *such that* $\overline{F}(n, m) \cap \overline{F}_i$ *consists of at most the element* n, *for each* $i = 1, 2, \cdots s$.

Theorem 1 follows from the following stronger result.

THEOREM 2. *If* v_1 *is a positive integer and* F *is a Fibonacci sequence, then for almost all* v_2, $\overline{F}(v_1, v_2) \cap \overline{F}$ *consists of at most the element* v_1.

Proof. Let the first two terms of F be u_1, u_2. Since

$$\lim_{k \to \infty} u_{k+1}/u_k = (1 + \sqrt{5})/2,$$

there is an integer m such that $u_{k+1}/u_k < 2$ for all $k \geq m$. Let n_0 be one such m for which, in addition, $u_{n_0+1} - u_{n_0} > v_1$ and $u_{n_0} > v_1$. We write

$$u_{n_0} = M_1, \ u_{n_0+1} = M_2, \ \cdots, \ u_{n_0+j} = M_{j+1}.$$

Thus $F(M_1, M_2) = F(u_{n_0}, u_{n_0+1})$.

We shall determine an upper bound L for the number of v_2's $(M_1 \leq v_2 < M_2)$ such that $\overline{F}(v_1, v_2)$ meets $\overline{F}(M_1, M_2)$. This L will turn out to be small in comparison with $M_1 - M_2$, the number of v_2's for which $M_1 \leq v_2 < M_2$. From this, Theorem 2 follows easily.

First let us examine where $\overline{F}(v_1, v_2)$ might meet $\overline{F}(M_1, M_2)$. Since $v_1 < M_1$ and $v_2 < M_2$, induction implies that $v_k < M_k$ for all $k \geq 1$. Thus if $v_k \in \overline{F}(M_1, M_2)$ and $v_k = M_i$, then i must be less than k. On the other hand, we shall show that $v_k > M_{k-2}$ for $k \geq 3$. Indeed, $v_3 > M_1$ and

$$v_4 = v_2 + v_3 > M_1 + M_1 > M_2.$$

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From the inequalities $v_3 > M_1$ and $v_4 > M_2$, induction yields the result:

$$v_k > M_{k-2} \qquad (k \geq 3).$$

Thus, if $v_k \in \overline{F}(M_1, M_2)$, then v_k must equal M_{k-1}. We now examine the v_2 for which $v_k = M_{k-1}$. Since $v_k = f_{k-1}v_2 + f_{k-2}v_1$, the equation $v_k = M_{k-1}$ is equivalent to

$$f_{k-1}v_2 + f_{k-2}v_1 = M_{k-1} \quad \text{or} \quad v_2 = \frac{M_{k-1} - f_{k-2}v_1}{f_{k-1}}.$$

Exploiting the fact that v_2 is to be an integer, we shall find the asserted bound L. To do this, we first define

$$x_k = \frac{M_{k-1} - f_{k-2}v_1}{f_{k-1}} \qquad (k \geq 3). \tag{1}$$

We shall show that the sequence $x_2, x_3, x_4, \cdots$ has a limit a, that it approaches this limit in an oscillating fashion, that is,

$$x_4 < x_6 < \cdots < a < \cdots < x_7 < x_5 < x_3,$$

and that $x_{k+1} - x_k$ approaches 0 quickly.

From (1) we obtain the equation

$$x_{k+1} - x_k = \frac{M_k - f_{k-1}v_1}{f_k} - \frac{M_{k-1} - f_{k-2}v_1}{f_{k-1}},$$

which reduces to

$$x_{k+1} - x_k = \frac{M_k f_{k-1} - M_{k-1}f_k - v_1(f_{k-1}^2 - f_k f_{k-2})}{f_k f_{k-1}}. \tag{2}$$

From the known identity of Tagiuri (see [2, vol. 1, p. 104 (reference 68)]),

$$f_r f_s - f_{r-1}f_{s+1} = (-1)^{r-1} f_i f_{i+s-r},$$

and the fact that $M_k = f_{k-1}M_2 + f_{k-2}M_1$, we obtain the equalities

$$f_{k-1}^2 - f_k f_{k-2} = (-1)^k \quad \text{and} \quad M_k f_{k-1} - M_{k-1}f_k = (-1)^k(M_2 - M_1).$$

Thus (2) becomes

$$x_{k+1} - x_k = \frac{(-1)^k(M_2 - M_1 - v_1)}{f_k f_{k-1}}. \tag{3}$$

Since $M_2 - M_1 > v_1$, (3) implies that the x_k's converge to a limit in the oscillating manner described above.

Letting $b = (1 + \sqrt{5})/2$, we see that $f_k \geq b^{k-2}$ $(k \geq 2)$, and thus, from (3),

$$|x_{k+1} - x_k| < \frac{M_2 - M_1 - v_1}{b^{2k-3}}.$$

Thus at most one x_k is an integer for those k satisfying the inequality

$$\frac{M_2 - M_1 - v_1}{b^{2k-3}} < 1$$

or, equivalently, the inequality

$$k > \frac{3 + \log_b (M_2 - M_1 - v_1)}{2}.$$

Thus the total number of k's for which x_k is an integer is at most

$$\frac{3 + \log_b (M_2 - M_1 - v_1)}{2}, \tag{4}$$

which is small in comparison with $M_2 - M_1$. Thus the number of v_2's $(M_1 \leq v_2 < M_2)$ for which $\overline{F}(v_1, v_2)$ meets $\overline{F}(M_1, M_2)$ is small in comparison with $M_2 - M_1$.

The same reasoning applies to the choice $M_1 = u_{n_0+1}$, $M_2 = u_{n_0+2}$ or to the choice $M_1 = u_{n_0+2}$, $M_2 = u_{n_0+3}$, and so on. From this and the fact that

$$\lim_{x \to \infty} (\log_b x)/x = 0,$$

Theorem 2 follows.

Remark 1. The proof of Theorem 2 shows that the v_2's for which

$$\overline{F}(v_1, v_2) \cap F(M_1, M_2) \neq \emptyset$$

tend to be close to each other. Specifically, by (1) and the fact that $b^2 = b + 1$,

$$\lim_{k \to \infty} x_k = M_1 + (b - 1)(M_2 - M_1 - v_1).$$

As an example, take $v_1 = 1$, $M_1 = 100$, and $M_2 = 161$. If $v_2 = 100$, then obviously $v_2 = M_1$. If $v_2 = 160$, then $v_3 = M_2 = 160$. If $v_2 = 130$, then $v_4 = M_3 = 261$. If $v_2 = 140$, then $v_5 = M_4 = 422$. If $v_2 = 136$, then $v_6 = M_5 = 683$. Also $x_7 = 137\frac{1}{2}$, $x_8 = 136\frac{12}{13}$, $x_9 = 137\frac{1}{7}$, and $x_{10} = 137\frac{1}{21}$. Thus 100, 160, 130, 140, and 136 are the only values of v_2 $(100 \leq v_2 < 161)$ for which

$$F(1, v_2) \cap F(100, 161) \neq \emptyset.$$

Also, since $[(3 + \log_b (161 - 100 - 1))/2] = 5$, the bound (4) can be fairly close.

Remark 2. If n is the smallest positive integer not in $\overline{F}_1, \overline{F}_2, \cdots, \overline{F}_s$, then, by Theorem 1, there is an F_{s+1}, containing n, and disjoint from each F_i $(i = 1, 2, \cdots, s.)$ From this we see that the set of positive integers is the disjoint union of Fibonacci sequences.

We next consider the case in which two Fibonacci sequences meet more than twice.

THEOREM 3. *Two Fibonacci sequences that meet at least three times are identical from some terms on.*

Proof. Assume that $\overline{F}(u_1, u_2) \cap \overline{F}(v_1, v_2) \neq \emptyset$ and that, in fact, they share a term which is not the initial term of either, that is, assume there exist integers i and j such that $u_i = v_j$, $i > 1$, $j > 1$. If $u_{i-1} = v_{j-1}$, then the two sequences are identical from some terms on. Let us assume that $u_{i-1} > v_{j-1}$. Then $u_{i+1} > v_{j+1}$. Also, $v_{j+2} > u_{i+1}$ since

$$v_{j+2} = v_{j+1} + v_j > u_i + u_{i-1} = u_{i+1}.$$

Thus,

$$u_i < v_{j+1} < u_{i+1}, \quad \text{and} \quad u_{i+1} < v_{j+2} < u_{i+2},$$

and, by induction,

$$u_{j+r-1} < v_{j+r} < u_{j+r} \qquad (r \geq 1).$$

Thus no v_{j+r} $(r \geq 1)$ is a term of $F(u_1, u_2)$. Theorem 3 follows from this.

Remark 3. Inspection of the proof of Theorem 3 shows that if $F(u_1, u_2)$ meets $F(v_1, v_2)$ exactly twice, then at least one of these two statements holds:

$$u_1 \in \overline{F}(v_1, v_2), \quad v_1 \in \overline{F}(u_1, u_2).$$

REFERENCES

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