ARITHMETICAL NOTES, VI. SIMULTANEOUS BINARY COMPOSITIONS INVOLVING COPRIME PAIRS OF INTEGERS

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1. INTRODUCTION

Let m, n denote positive integers, and let $Q \equiv Q(m, n)$ represent the number of sets of integers x_1, x_2, y_1, y_2 such that

(1.1) $$m = x_1 + y_1, \quad n = x_2 + y_2, \quad x_j > 0, \; y_j > 0 \qquad (j = 1, 2),$$

subject to the restriction

(1.2) $$(x_1, x_2) = (y_1, y_2) = 1.$$

It is the object of this note to prove

THEOREM A. *If* $n \geq m$, *then*

(1.3) $$Q(m, n) \sim mn\,\alpha(m, n) \qquad as\ m \to \infty,$$

where

(1.4) $$\alpha(m, n) = \prod_{p \mid (m,n)} \left(1 - \frac{1}{p^2}\right) \prod_{p \nmid (m,n)} \left(1 - \frac{2}{p^2}\right).$$

(Throughout this note, p stands for a prime.)

As a consequence of this result, one may obtain

THEOREM B. *There exist positive constants* A_1, A_2 *such that, when* m *and* n *are sufficiently large,*

$$A_1 < Q(m, n)/mn < A_2.$$

Theorem A is actually proved in a slightly stronger form (see Theorem 3.1). The proof is based on an elementary method similar to that employed by Mirsky in [3].

2. SOME LEMMAS

Let $\theta_p(m, n)$ denote the number of solutions (mod p) of

$$m \equiv x_1 + y_1, \; n \equiv x_2 + y_2 \pmod{p}, \qquad p \nmid (x_1, x_2), \; p \nmid (y_1, y_2).$$

The following result is the special case $r = p$ of [1, (8.8), Corollary 18.1].

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LEMMA 2.1.

$$\theta_p(m, n) = \begin{cases} p^2 - 1 & \text{if } p \mid (m, n), \\ p^2 - 2 & \text{if } p \nmid (m, n). \end{cases} \tag{2.1}$$

LEMMA 2.2 *There exist positive constants* c_1, c_2 *such that*

$$c_1 \leq \alpha(m, n) \leq c_2 \tag{2.2}$$

for all m, n.

Proof. In fact, by (1.4),

$$\prod_p \left(1 - \frac{2}{p^2}\right) \leq \alpha(m, n) \leq \prod_p \left(1 - \frac{1}{p^2}\right).$$

LEMMA 2.3. *If* $\Omega(n)$ *denotes the number of prime-power divisors of* n, *including the primes, then, for all* $\delta \geq \log 2/\log 3$,

$$2^{\Omega(n)} = O(n^\delta) \qquad (n \in S), \tag{2.3}$$

where S *is the sequence of odd integers.*

Proof. The function $f_\delta(n) = 2^{\Omega(n)}/n^\delta$ is clearly multiplicative on S. It follows that $f_\delta(n)$ is bounded on S, because $f_\delta(p^m) = 2^m/p^{\delta m} \leq 1$ for odd primes p.

Finally, we note the following refinement of Chebyshev's Theorem.

LEMMA 2.4 (compare [2, Section 22.2, Theorem 415]). *For all* $n \geq 1$,

$$\sum_{p \leq n} \log p < n \log 4. \tag{2.4}$$

3. THE PROOF

We prove the following slightly refined form of Theorem A.

THEOREM 3.1. *For* $2 \leq m \leq n$,

$$Q(m, n) = mn \left\{ \alpha(m, n) + O\left(\frac{1}{\sqrt[4]{\log m}}\right) \right\}. \tag{3.1}$$

Proof. In the proof, the ith prime will be denoted by p_i. The quantity $x = x(m)$ will denote a function of m to be determined later, subject to the condition

$$3 \prod_{p \leq n} p < m \qquad (x \geq 2), \tag{3.2}$$

with m supposed sufficiently large, let us say, $m \geq e^6$. We shall use $k = k(m)$ to denote the largest integer i such that $p_i \leq x$; moreover, it will be convenient to write $r_k = p_1 \cdots p_k$.

Let $Q_1 = Q_1(m, n)$ denote the number of solutions of (1.1) such that neither (x_1, x_2) nor (y_1, y_2) is divisible by any $p \leq x$. Let $Q_2 = Q_2(m, n)$ denote the number of such solutions of (1.1), with the proviso that either (x_1, x_2) or (y_1, y_2) is divisible by at least one $p > x$. It follows then that

$$Q = Q_1 - Q_2. \tag{3.3}$$

First we estimate Q_2. Evidently,

$$Q_2 \leq \sum_{p>x} \sum_{\substack{(1.1) \\ p|(x_1,x_2) \text{ or} \\ p|(y_1,y_2)}} 1 \leq 2 \sum_{p>2} \sum_{\substack{(1.1) \\ p|(x_1,x_2)}} 1,$$

the summation symbols being self-explanatory. Hence

$$Q_2 = O\left(\sum_{p>x} \sum_{\substack{m=px_1+y_1 \\ n=px_2+y_2}} 1 \right) = O\left(\sum_{p>x} \frac{m}{p} \cdot \frac{n}{p} \right) = O\left(mn \sum_{t>x} \frac{1}{t^2} \right),$$

so that

$$Q_2 = O\left(\frac{mn}{x} \right). \tag{3.4}$$

In estimating Q_1, it is useful to note that if $\{x_1, x_2, y_1, y_2\}$ is a solution of (1.1), then the condition

$$p_i \nmid (x_1, x_2), \quad p_i \nmid (y_1, y_2) \qquad (1 \leq i \leq k), \tag{3.5}$$

is satisfied if and only if there exist ξ_{ij}, η_{ij} such that

$$0 \leq \xi_{ij} < p_i, \quad 0 \leq \eta_{ij} < p_i \qquad (1 \leq i \leq k,\ j = 1, 2), \tag{3.6}$$

$$\xi_{i1} + \eta_{i1} \equiv m, \quad \xi_{i2} + \eta_{i2} \equiv n \pmod{p_i} \qquad (1 \leq i \leq k), \tag{3.7}$$

$$p_i \nmid (\xi_{i1}, \xi_{i2}), \quad p_i \nmid (\eta_{i1}, \eta_{i2}) \qquad (1 \leq i \leq k), \tag{3.8}$$

and

$$x_j \equiv \xi_{ij}, \quad y_j \equiv \eta_{ij} \pmod{p_i} \qquad (1 \leq i \leq k,\ j = 1, 2). \tag{3.9}$$

Thus we have

$$Q_1 = \sum_{\substack{p_i \\ (1.1),(3.5)}} 1 = \sum_{\xi_{ij}, \eta_{ij}}^{*} \sum_{\substack{x_j, y_j \\ (1.1),(3.9)}} 1, \tag{3.10}$$

where the * indicates that ξ_{ij} *and* η_{ij} *satisfy* (3.6),(3.7), *and* (3.8). Corresponding to each pair ξ_{ij}, η_{ij} in (3.6) there exist, by the Chinese Remainder Theorem, uniquely determined $\xi_j, \eta_j \pmod{r_k}$ such that

(3.11) $$\xi_j \equiv \xi_{ij},\ \eta_j \equiv \eta_{ij} \pmod{r_k} \qquad (j = 1, 2);$$

moreover, one may suppose that

(3.12) $$0 \le \xi_j < r_k, \quad 0 \le \eta_j < r_k \qquad (j = 1, 2).$$

If, in addition, (3.7) is satisfied, then it follows that

(3.13) $$m \equiv \xi_1 + \eta_1, \quad n \equiv \xi_2 + \eta_2 \pmod{r_k}.$$

In view of (3.11), the condition (3.9) may be replaced by

(3.14) $$x_j \equiv \xi_j, \quad y_j \equiv \eta_j \pmod{r_k} \qquad (j = 1, 2).$$

Hence, if we write

(3.15) $$x_j = \xi_j + r_k X_j, \quad y_j = \eta_j + r_k Y_k \qquad (j = 1, 2),$$

then (1.1) is replaced by

(3.16) $$X_1 + Y_1 = \frac{m - \xi_1 - \eta_1}{r_k}, \qquad X_2 + Y_2 = \frac{n - \xi_2 - \eta_2}{r_k},$$

where X_j and Y_j are integers and

(3.17) $$X_j \ge 0,\ Y_j \ge 0,\ X_j > 0 \text{ if } \xi_j = 0,\ Y_j > 0 \text{ if } \eta_j = 0 \quad (j = 1, 2).$$

The assumption that $m > 3r_k$, in connection with (3.12) and (3.13), ensures that (3.16) is solvable subject to (3.17). Hence, by (3.10) and (3.15),

$$Q_1 = \sum_{\xi_{ij},\eta_{ij}}^{*} \sum_{\substack{X_j, Y_j \\ (3.16),(3.17)}} 1$$

$$= \sum_{\xi_{ij},\eta_{ij}}^{*} \left(\frac{m - \xi_1 - \eta_1}{r_k} + O(1)\right)\left(\frac{n - \xi_2 - \eta_2}{r_k} + O(1)\right),$$

so that, by (3.12),

$$Q = \sum_{\xi_{ij},\eta_{ij}}^{*} \frac{mn}{r_k^2} + O\left(\sum_{\xi_{ij},\eta_{ij}}^{*} \frac{n}{r_k}\right) + O(1)$$

$$= mn \prod_{p \le x} \frac{\theta_p(m, n)}{p^2} + O\left(n \prod_{p \le x} \frac{\theta_p(m, n)}{p}\right) + O(1).$$

Since obviously $\theta_p(m, n) \le p^2$, we have

(3.18) $$Q_1 = mnQ^* + O\left(n \prod_{p \le x} p\right), \qquad Q^* = \prod_{p \le x} \frac{\theta_p(m, n)}{p^2}.$$

By (2.4),

(3.19) $$\prod_{p\leq x} p \leq e^{2x},$$

so that (3.18) becomes

(3.20) $$Q_1 = mnQ^* + (n e^{2x}).$$

Application of Lemmas 2.1 and 2.2 yields

$$Q^* = \alpha(m, n) \prod_{\substack{p>x\\ p\mid(m,n)}} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{\substack{p>x\\ p\nmid(m,n)}} \left(1 - \frac{2}{p^2}\right)^{-1}$$

$$= \alpha(m, n) \prod_{\substack{p>x\\ p\mid(m,n)}} \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots\right) \prod_{\substack{p>x\\ p\nmid(m,n)}} \left(1 + \frac{2}{p^2} + \frac{2^2}{p^4} + \frac{2^3}{p^6} + \cdots\right)$$

$$= \alpha(m, n) \left\{1 + O\left(\sum_{\substack{r,s\\ rs>1}}{}' \frac{2^{\Omega(s)}}{r^2 s^2}\right)\right\},$$

where in the Σ' summation, $p \mid r \rightarrow p > x$, and $p \mid s \rightarrow p > x$. Therefore, by Lemma 2.2,

$$Q^* = \alpha(m, n) + O\left(\sum_{\substack{r,s\\ rs>1}}{}' \frac{2^{\Omega(s)}}{r^2 s^2}\right)$$

$$= \alpha(m, n) + O\left(\sum_{r>x}{}' \frac{1}{r^2}\right) + O\left(\sum_{s>x}{}' \frac{2^{\Omega(s)}}{s^2}\right) + O\left(\sum_{s>x}{}' \frac{2^{\Omega(s)}}{s^2} \sum_{r>x} \frac{1}{r^2}\right),$$

and hence

(3.21) $$Q^* = \alpha(m, n) + O\left(\sum_{s>x}{}' \frac{2^{\Omega(s)}}{s^2}\right).$$

The s-summation in (3.21) is restricted to *odd* s, because $x \geq 2$; thus, Lemma 2.3 is applicable, whereby

$$Q^* = \alpha(m, n) + O\left(\sum_{s>x} \frac{1}{s^{2-\delta}}\right) \qquad (1 > \delta \geq \log 2/\log 3).$$

In particular, with $\delta = 3/4$,

(3.22) $$Q^* = \alpha(m, n) + O\left(\frac{1}{\sqrt[4]{x}}\right),$$

so that by (3.20), (3.3), and (3.4),

$$Q = mn\alpha(m, n) + O\left(\frac{mn}{\sqrt[4]{x}}\right) + O(n e^{2x}). \tag{3.23}$$

We choose now $x = (\log m)/3$ $(m \geq e^6)$, noting that (3.2) is satisfied by virtue of (3.19). The relation (3.1) now follows, under the condition $m \geq e^6$. But on the basis of Lemma 2.2, (3.1) is valid for $2 \leq m \leq n$, in the form $Q(m, n) = O(n)$. Hence the theorem is proved.

On the basis of (2.2), Theorem A is a consequence of Theorem 3.1, while Theorem B follows from Theorem A.

Remark (added May 29, 1962). It is easily observed that Theorems A and B remain valid if $Q(m, n)$ is replaced by $Q'(m, n)$, where $Q'(m, n)$ is defined to be the number of solutions of (1.1) and (1.2) with $x_j > 1$, $y_j > 1$.

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