ON MODULAR FORMS OF LEVELS TWO AND THREE

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1. We consider the problem of parametrizing all modular forms of dimension r on certain subgroups of the modular group, namely, on the principal congruence subgroups of levels two and three. As a consequence of the parametrization of the forms, all multiplier systems for the dimension r are determined. We rely on the results of Maak [4] for the determination of all one-dimensional representations for the principal congruence subgroup of level 2. We then use his method to determine the characters on the principal congruence subgroup of level 3.

The parametrization of modular forms for the full modular group has been carried out by Rademacher and Zuckerman [8], [9]. We use arguments similar to those of Section 8 of [9].

2. The homogeneous modular group $\overline{\Gamma}(1)$ consists of all 2-by-2 matrices $V = (a\ b \mid c\ d)$ (written in one line) with rational integral entries and determinant 1. To each element of this group there corresponds a modular substitution

$$Vz = \frac{az + b}{cz + d}.$$

Note that V and $-V = (-a, -b \mid -c, -d)$ correspond to the same substitution. We denote the group of substitutions by $\Gamma(1)$. This group is known to be generated by

$$Sz = z + 1 \qquad \text{and} \qquad Tz = -1/z.$$

The principal congruence subgroup of level N, $\overline{\Gamma}(N)$, consists of all those $V \in \overline{\Gamma}(1)$ for which $V \equiv \pm I \pmod N$, where $I = (1\ 0 \mid 0\ 1)$, and where we mean element-wise congruence. We shall be interested in cases where N = 2, 3.

A fundamental region for the modular group is the set

$$R(1) = \{z = x + iy;\ |z| > 1,\ |x| < 1/2,\ y > 0\}.$$

A fundamental region for the group $\Gamma(N)$ is the set

$$\bigcup_{k=1}^{\mu} V_k R(1),$$

where $V_1, \cdots, V_\mu$ are the representatives of the coset decomposition of $\Gamma(1)$ modulo $\Gamma(N)$. Let r be a real number. A modular form of dimension r for the group $\Gamma(N)$ is a meromorphic function F(z) in the upper half-plane which satisfies

$$F(Vz) = v(V)\,(cz + d)^{-r}\,F(z)$$

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for each $V = (a\ b \mid c\ d) \in \Gamma(N)$. Here $v(V)$ denotes a complex number of absolute value 1 which, in the case r is integral, is a character of the group. Otherwise, v satisfies a certain "consistency condition" (see for example [3, p. 73] where the notation is slightly different). It is further assumed that $F(z)$ has a finite number of poles (measured in the appropriate local variable) within the closure of the fundamental region for $\Gamma(N)$.

The function

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}) \qquad (\Im z > 0)$$

is a modular form of dimension $-1/2$ for the full modular group, with no zeros in the upper half-plane. Therefore, $h(z) = \log \eta(z)$ can be defined as a single-valued analytic function in the upper half-plane. We take the principal branch of the logarithm throughout this discussion. The behavior of $h(z)$ under modular substitutions is known (for a recent short proof, see [6]). We list these results in the form in which we shall use them, which is slightly different from that given in [6]. For $V = (a\ b \mid c\ d) \in \Gamma(1)$,

$$\log \eta(Vz) = \log \eta(z) + (\text{sign } c)^2 \frac{1}{2} \log (cz + d) - (\pi i/4) \text{ sign } c + \pi i \Phi(V)/12, \tag{2.1}$$

with $-\pi < \Im \log (cz + d) < \pi$ for $c \neq 0$ and

$$\text{sign } c = \begin{cases} c/|c| & \text{if } c \neq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{2.2}$$

Furthermore,

$$\Phi(V) = \Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} b/d & \text{if } c = 0, \\ (a + d)/c + 12(\text{sign } c)\, s(a, |c|) & \text{if } c \neq 0, \end{cases} \tag{2.3}$$

where $s(h, k)$ is the Dedekind sum

$$s(h, k) = \sum_{\mu=1}^{k} (\mu/k - [\mu/k] - 1/2)(\mu h/k - [\mu h/k] - 1/2) \tag{2.4}$$

and the symbol $[u]$ represents the greatest-integer function.

3. $\Gamma(2)$ is a free group of rank 2; its generators are

$$S^2 z = z + 2, \qquad T^{-1} S^2 T z = \frac{z}{-2z + 1}.$$

Suppose that $F(z)$ is a modular form of dimension r for $\Gamma(2)$, with the multiplier system v. Then $H(z) = F(z)\eta^{2r}(z)$ is a modular form of dimension 0 with a multiplier system v'. But this means that v' is a character on $\Gamma(2)$. All characters on $\Gamma(2)$ have been determined by Maak [4]. We outline his results, since we shall need them later. If α and β are real and

(3.1) $$h_{\alpha\beta}(z) = \eta^{\alpha}(2z)\,\eta^{\beta}(z/2)\,\eta^{-(\alpha+\beta)}((z+1)/2),$$

then

(3.2) $$h_{\alpha\beta}(Vz) = v(V)\,h_{\alpha\beta}(z), \qquad v(V) = \exp\{\pi i \Psi_{\alpha\beta}(V)/12\},$$

where

(3.3) $$\begin{aligned}\Psi_{\alpha\beta}(V) &= \Psi_{\alpha\beta}\begin{pmatrix} a & b \\ c & d\end{pmatrix} \\ &= \alpha\Phi\begin{pmatrix} a & 2b \\ c/2 & d\end{pmatrix} + \beta\Phi\begin{pmatrix} a & b/2 \\ 2c & d\end{pmatrix} - (\alpha+\beta)\Phi\begin{pmatrix} a+c & (b+d-a-c)/2 \\ 2c & d-c\end{pmatrix};\end{aligned}$$

furthermore,

(3.4) $$\Psi_{\alpha\beta}\begin{pmatrix} 1 & 2 \\ 0 & 1\end{pmatrix} = 3\alpha, \quad \Psi_{\alpha\beta}\begin{pmatrix} 1 & 0 \\ -2 & 1\end{pmatrix} = 3\beta.$$

Thus $H(z)$ has one of the multiplier systems given above. We suppose that

(3.5) $$H(z+2) = e^{2\pi i\alpha}H(z), \qquad H(z/(-2z+1)) = e^{2\pi i\beta}H(z),$$

$0 \leq \alpha,\ \beta < 1$. Since $\Gamma(2)$ is a group of genus 0, that is, since its fundamental region has genus 0, it has an invariant function $\lambda(z)$ that maps the fundamental region one-to-one and conformally onto the plane (see [1] for the important properties of $\lambda(z)$). For definiteness, we take a particular fundamental region with cusps at 0, 1, $i\infty$; its other properties will not matter. The function $\lambda(z)$ has a zero at the cusp $i\infty$, and a pole at $z = 1$. The function $\eta(z)$ has zeros at the cusps of the fundamental region. We see that $H(z)$ has the same zeros and poles as $F(z)$, with the exception of the cusps. Let the zeros be at $\sigma_1, \cdots, \sigma_Z$, the poles at $\rho_1, \cdots, \rho_P$; define

$$\Theta(z) = \prod_{j=1}^{P}(\lambda(z) - \lambda(\rho_j))\prod_{k=1}^{Z}(\lambda(z) - \lambda(\sigma_k))^{-1},$$

and let $G(z) = H(z)\,\Theta(z)$. Then $G(z)$ has no zeros or poles in the fundamental region, except possibly at the cusps; it is a modular form of dimension 0 for $\Gamma(2)$; and it satisfies the same functional equations (3.5) as does $H(z)$. In view of (3.5), $G(z)$ has at the cusps the expansions (see for example [5, p. 493])

(3.6) $$G(z) = \sum_{n=m_1}^{\infty} a_n e^{2\pi i(\alpha+n)z/2},$$

(3.7) $$G_T(z) = \sum_{n=m_2}^{\infty} b_n e^{2\pi i(\beta+n)z/2},$$

(3.7) $$G_A(z) = \sum_{n=m_3}^{\infty} c_n e^{2\pi i(\gamma+n)z/2} \qquad (A = (ST)^{-1}),$$

where $G_V(z) = G(z) \mid V^{-1} = G(V^{-1} z)$. Since $G(z)$ has no poles in the fundamental region, except possibly at the cusps, integration of G'/G around the boundary of the fundamental region of $\Gamma(2)$ modified by small circular detours at the cusps gives the relation

$$m_1 + \alpha + m_2 + \beta + m_3 + \gamma = 0 \tag{3.9}$$

as the circular detours are allowed to approach their points.

Write

$$\alpha' = -8(m_1 + \alpha), \qquad \beta' = -8(m_2 + \beta), \tag{3.10}$$

and consider the function $h_{\alpha'\beta'}(z)$ defined in (3.1). Using (3.4), we see that

$$h_{\alpha'\beta'}(z + 2) = e^{-2\pi i\alpha} h_{\alpha'\beta'}(z), \tag{3.11}$$

$$h_{\alpha'\beta'}(z/(-2z + 1)) = e^{-2\pi i\beta} h_{\alpha'\beta'}(z). \tag{3.12}$$

Furthermore, using the known expansions for $\eta(z)$ and the relations

$$h_{\alpha\beta}(z) \mid T = v_1 2^{(\beta-\alpha)/2} h_{\beta\alpha}(z), \qquad |v_1| = 1,$$

we find that

$$h_{\alpha'\beta'}(z) = \exp[-2\pi i(\alpha + m_1)z/2] + \cdots, \tag{3.13}$$

$$h_{\alpha'\beta'}(z) \mid T = v_1 2^{(\beta'-\alpha')/2} \exp[-2\pi i(\beta + m_2)z/2] + \cdots. \tag{3.14}$$

Since $h_{\alpha'\beta'}(z)$ is a form of dimension 0 on $\Gamma(2)$, it too satisfies a relation like (3.9), indeed, the same relation. Now define $K(z) = G(z) h_{\alpha'\beta'}(z)$. Comparing (3.5) with (3.11) and with (3.12), we see that $K(z)$ is invariant with respect to the substitutions of $\Gamma(2)$. Also, comparing the expansions (3.6), (3.7) and (3.8) with (3.13) and (3.14) and taking into account the relation (3.9) satisfied by both $G(z)$ and $h_{\alpha'\beta'}(z)$, we see that $K(z)$ has no poles in the closed fundamental region for $\Gamma(2)$. Thus, $K(z)$ is a constant K. We obtain

$$\begin{aligned} F(z) &= K\eta^{-2r}(z)\,\eta^{-\alpha'}(2z)\,\eta^{-\beta'}(z/2)\,\eta^{(\alpha'+\beta')}((z + 1)/2) \\ &\quad \cdot \prod_{j=1}^{Z} (\lambda(z) - \lambda(\sigma_j)) \prod_{k=1}^{P} (\lambda(z) - \lambda(\rho_k))^{-1}. \end{aligned} \tag{3.15}$$

Thus we have proved the following proposition.

THEOREM 1. *The modular form* (3.15) *of dimension* r *satisfies the relation*

$$F(Vz) = v(V)\,(cz + d)^{-r}\, F(z) \qquad (V = (a\ b \mid c\ d) \in \overline{\Gamma}(2)),$$

where

$$v(V) = \exp[-\pi i r\,\Phi(V)/6 + \pi i r\,(\text{sign } c)/2 - \pi i \Psi_{\alpha'\beta'}(V)/12] \tag{3.16}$$

and $\Phi(V)$ *and* $\Psi_{\alpha'\beta'}(V)$ *are defined in* (2.3) *and* (3.3), *respectively. Conversely,* *every modular form of dimension* r *for* $\Gamma(2)$ *is of this type, with the multiplier system given above. The* $\lambda(z)$ *appearing in* (3.15) *is the modular invariant on* $\Gamma(2)$, *it takes each value once, and it has a pole at* 1 *and a zero at* $i\infty$.

4. We outline the discussion for the case of $\Gamma(3)$, which is a free group of rank three and with the generators

$$S^3 z = z + 3, \quad T^{-1}S^3 Tz = \frac{z}{-3z+1}, \quad S^{-1}T^{-1}S^3 TSz = \frac{4z+3}{-3z-2}.$$

First we compute all characters for $\Gamma(3)$. Following Maak [4], we write

$$\begin{aligned}\log h_{\alpha\beta\gamma}(z) &= \alpha \log \eta(3z) + \beta \log \eta(z/3) + \gamma \log \eta((z+1)/3) \\ &\quad - (\alpha+\beta+\gamma)\log \eta((z+2)/3),\end{aligned} \tag{4.1}$$

where the principal branch of the logarithm is meant. By (2.1),

$$\log h_{\alpha\beta\gamma}(Vz) = \pi i \Psi_{\alpha\beta\gamma}(V)/12 + \log h_{\alpha\beta\gamma}(z), \tag{4.2}$$

where

$$\begin{aligned}\Psi_{\alpha\beta\gamma}(V) &= \alpha\Phi\begin{pmatrix} a & 3b \\ c/3 & d\end{pmatrix} + \beta\Phi\begin{pmatrix} a & b/3 \\ 3c & d\end{pmatrix} + \gamma\Phi\begin{pmatrix} a+c & (b+d-a-c)/3 \\ 3c & d-c\end{pmatrix} \\ &\quad - (\alpha+\beta+\gamma)\Phi\begin{pmatrix} a+2c & (b+2d-2a-4c)/3 \\ 3c & d-2c\end{pmatrix}.\end{aligned} \tag{4.3}$$

Thus,

$$h_{\alpha\beta\gamma}(Vz) = \exp[\pi i \Psi_{\alpha\beta\gamma}(V)/12]\, h_{\alpha\beta\gamma}(z). \tag{4.4}$$

It follows that $\exp[\pi i \Psi_{\alpha\beta\gamma}(V)/12]$ is a character on $\Gamma(3)$.

To verify that all characters are found in this way, we compute the values of $\Psi_{\alpha\beta\gamma}$ for the generators of $\Gamma(3)$:

$$\Psi_{\alpha\beta\gamma}(S^3) = 8\alpha, \tag{4.5a}$$

$$\Psi_{\alpha\beta\gamma}(T^{-1}S^3T) = 8\beta, \tag{4.5b}$$

$$\Psi_{\alpha\beta\gamma}(S^{-1}T^{-1}S^3TS) = 8\gamma. \tag{4.5c}$$

The computation is straight-forward; we can make it easier by using the following result of Rademacher [7, p. 463].

THEOREM. *If* $c > 0$ *and* $(d \pm 1)^2 \equiv 0 \bmod c$, *then*

$$s(d, c) = \mp \frac{\mu^2 + 2 - 3c}{12c},$$

with $\mu = (c, d \pm 1)$ *and with the upper or lower signs taken together.*

Next we choose for $\Gamma(3)$ a fundamental region that has cusps at $i\infty$, 0, +1, -1. The modular invariant $\xi(z)$ [3, pp. 611-613] that maps this fundamental region one-to-one and conformally onto the whole plane has a pole at $i\infty$ and a zero at $z = 0$ (we do not use the invariant ξ given in [3], but rather $\xi - 1$; this is for the technical reason that we desire the modular invariant to have its pole and zero at cusps).

Suppose that $F(z)$ is a modular form of dimension r on $\Gamma(3)$. Then $H(z) = F(z)\eta^{2r}(z)$ is a modular form of dimension 0 on $\Gamma(3)$. It has one of the multiplier systems characterized above. Let

$$H(z+3) = e^{2\pi i\alpha} H(z), \qquad H(z/(-3z+1)) = e^{2\pi i\beta} H(z),$$
$$H((4z+3)/(-3z-2)) = e^{2\pi i\gamma} H(z), \tag{4.6}$$

with $0 \leq \alpha, \beta, \gamma < 1$. As before, $F(z)$ has Z zeros at $\sigma_1, \cdots, \sigma_Z$ and P poles at $\rho_1, \cdots, \rho_P$, where the σ's and the ρ's are points of the closed fundamental region other than the cusps. We define $\Theta(z)$ as before, using $\xi(z)$ in place of $\lambda(z)$, and then set $G(z) = H(z)\,\Theta(z)$. The function $G(z)$ has a Fourier expansion in the local variable appropriate to the cusps $i\infty, 0, +1, -1$. Let the first nonvanishing term of these expansions have exponent $m_1 + \alpha$, $m_2 + \beta$, $m_3 + \gamma$, $m_4 + \delta$, respectively. As before,

$$m_1 + \alpha + m_2 + \beta + m_3 + \gamma + m_4 + \delta = 0. \tag{4.7}$$

Define

$$\alpha' = -3(m_1 + \alpha), \quad \beta' = -3(m_2 + \beta), \quad \gamma' = -3(m_3 + \gamma), \tag{4.8}$$

and consider $h(z) = h_{\alpha'\beta'\gamma'}(z)$. We find that

$$h(z+3) = e^{-2\pi i\alpha} h(z), \qquad h(z/(-3z+1)) = e^{-2\pi i\beta} h(z),$$
$$h((4z+3)/(-3z-2)) = e^{-2\pi i\gamma} h(z). \tag{4.9}$$

Further, when $h(z)$ is expanded in its Fourier series at the cusps $i\infty, 0, +1, -1$, the exponent of the first nonvanishing term is $-m_1 - \alpha$, $-m_2 - \beta$, $-m_3 - \gamma$, $-m_4 - \delta$, respectively. In this connection, it is helpful to note the relations

$$h_{\alpha\beta\gamma}(z)\,|\,T = v_1 3^{(\beta-\alpha)/2} h_{\beta\alpha\delta}(z) \qquad (|v_1| = 1),$$
$$h_{\alpha\beta\gamma}(z)\,|\,ST = v_2 3^{(\delta-\alpha)/2} h_{\gamma\alpha\delta}(z) \qquad (|v_2| = 1) \tag{4.10}$$

with $-\delta = \alpha + \beta + \gamma$. We see also that the first exponents for $h_{\alpha'\beta'\gamma'}(z)$ satisfy (4.7), so that we need not expand this function at -1 to deduce that the first exponent is $-m_4 - \delta$. Putting these facts together, we conclude that $h_{\alpha'\beta'\gamma'}(z)\,G(z)$ is a constant K. Hence

$$F(z) = K\eta^{-2r}(z)\,\eta^{-\alpha'}(3z)\,\eta^{-\beta'}(z/3)\,\eta^{-\gamma'}((z+1)/3)\,\eta^{\delta'}((z+2)/3)$$
$$\cdot \prod_{J=1}^{Z} (\xi(z) - \xi(\sigma_j)) \prod_{k=1}^{P} (\xi(z) - \xi(\rho_k))^{-1}, \tag{4.11}$$

with $\delta' = \alpha' + \beta' + \gamma'$.

THEOREM 2. *The modular form* (4.11) *of dimension* r *satisfies*

$$F(Vz) = v(V)(cz + d)^{-r} F(z) \qquad (V = (a\ b \mid c\ d) \in \bar{\Gamma}(3)),$$

where

$$v(V) = \exp[-\pi i r \Phi(V) + \pi i r (\text{sign } c)/2 - \pi i \Psi_{\alpha'\beta'\gamma'}(V)/12] \tag{4.12}$$

and $\Phi(V)$ *and* $\Psi_{\alpha'\beta'\gamma'}(V)$ *are defined in* (2.3) *and* (4.3), *respectively. Conversely, every modular form is of this form. The function* $\xi(z)$ *is the modular invariant for* $\Gamma(3)$; *it takes each value once, and it has a pole at* $i\infty$ *and a zero at* $z = 0$.

Remarks. The methods given above seem to work also for $\Gamma(4)$; however, the details are not so neat. The principal congruence group of level five, $\Gamma(5)$, is also of genus 0 and therefore possesses a modular invariant which maps its fundamental region one-to-one and conformally onto the plane. This group is free and has rank eleven. There exist only six "transformations of order 5" that are pairwise left-inequivalent with respect to $\Gamma(1)$, namely $5z$, $z/5$, $(z + 1)/5$, $\cdots$, $(z + 4)/5$. Thus, it is not clear how one can define a function with the properties of $h_{\alpha\beta}(z)$. It seems that in the case of modular forms of levels greater than four, new methods will be needed for the determination of characters and for the parametrization of modular forms on the principal congruence subgroups.

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