A NOTE ON THE HOMOLOGY GROUPS OF RELATIONS

John Thomas

The purpose of this paper is to give a new proof of Dowker's theorem that "related" homology and cohomology theories are isomorphic. In [1], Dowker observed that a relation between two sets X and Y endows each with the structure of a simplicial complex, and he showed that the homology groups of X and Y derived from these simplicial structures are isomorphic. The isomorphism constructed by Dowker seems rather complicated, and it is not immediately apparent that it is natural (as must be verified for Dowker's main application) that the Čech and Vietoris homology theories and the Čech and Alexander cohomology theories are isomorphic, when they are based on the same family of coverings.

In this paper we exhibit an isomorphism that is somewhat simpler, and (granted a certain amount of algebraic machinery) more clearly natural. Roughly speaking, we proceed as follows. We consider the category of relation pairs and their maps, and define three homology functors on this category—the two that Dowker considered, and a third, which is defined in terms of a certain double complex; and we show that all three functors are naturally isomorphic.

Here is a brief sketch of the isomorphism between the Čech and Vietoris homology theories. Consider the category of pairs (X, α) in which X is a topological space and α is a cover of X taken from a particular directed family of coverings, say $\Omega(X)$. The maps $(f, \phi)\colon (X, \alpha) \to (Y, \beta)$ are pairs in which f is continuous and ϕ is a function. The relation on $X \times \alpha$ is the inclusion relation: x in X and O in α are related if $x \in O$. Let the homology groups induced by the two simplicial structures be denoted by ${}^1H(X, \alpha)$ and ${}^2H(X, \alpha)$. The maps induced by

$$(1, \phi)\colon (X, \alpha) \to (X, \beta),$$

where α refines β and ϕ is relation preserving, give inverse systems for each X, and as Dowker pointed out,

$$\lim_{\alpha\in\Omega(X)} \text{inv}\ {}^1H(X, \alpha) \quad \text{and} \quad \lim_{\alpha\in\Omega(X)} \text{inv}\ {}^2H(X, \alpha)$$

are the Vietoris and Čech groups, respectively. Since the functors ${}^1H$ and ${}^2H$ are isomorphic, they are still isomorphic when composed with the functor lim inv, whence the Čech and Vietoris homology theories are isomorphic.

1. DOUBLE COMPLEXES

In this section we recall some of the basic facts about double complexes. Let A be a principal ideal domain. A *double complex* K over A consists of a bigraded left A-module $K = \Sigma K_{p,q}$ (p, q integers), together with a pair of homogeneous endomorphisms

Received January 12, 1962.

The result of this paper is a minor part of my doctoral dissertation at the University of Oklahoma, June 1959. I wish to express my gratitude to Professor John Giever for his help and encouragement during the preparation of that thesis.

$$d_1\colon K_{p,q} \to K_{p+r,q} \quad \text{and} \quad d_2\colon K_{p,q} \to K_{p,q+r}$$

satisfying

$$d_1 \cdot d_1 = d_2 \cdot d_2 = d_1 \cdot d_2 + d_2 \cdot d_1 = 0\,.$$

For our purposes, r will always be $+1$ or -1. If $r = -1$, we shall call it a double chain complex, if $r = +1$, a double cochain complex; in the latter case we shall, as is customary, write the indices of the bigradation as superscripts rather than as subscripts. In the following discussion we shall generally pretend that we are dealing with chain complexes, letting the reader supply the analogous statements for cochain complexes.

A *subcomplex* L is a bigraded submodule that is stable under the action of d_1 and d_2, that is,

$$d_1(L_{p,q}) \subset L_{p+r,q} \quad \text{and} \quad d_2(L_{p,q}) \subset L_{p,q+r}\,.$$

It is easy to verify that ${}^kZ(K) = \operatorname{Ker} d_k$ and ${}^kB(K) = \operatorname{Im} d_k$ $(k = 1, 2)$ are subcomplexes. If L is a subcomplex of K, we may also form the quotient double complex K/L, bigraded by $(K/L)_{p,q} = K_{p,q}/L_{p,q}$ with endomorphisms d_1 and d_2 induced from K. In particular, ${}^kH(K) = {}^kZ(K)/{}^kB(K)$ $(k = 1, 2)$ are quotient double complexes.

A *map* of double complexes is a module homomorphism of bidegree $(0, 0)$ that commutes with d_1 and d_2. The collection of double complexes and their maps forms a category, which we denote by $\mathscr{D}$. We shall in fact be interested only in a subcategory $\mathscr{N}$ of *regular* double complexes that satisfy the additional conditions

(1) $K_{p,q} = 0$ if $p < 0$ or $q < 0$ (K is properly bigraded),

(2) $K_{p,q}$ is a free module for every pair p, q,

(3) ${}^1H_{p,q}(K) = 0$ for $q > 0$ and ${}^2H_{p,q}(K) = 0$ for $p > 0$.

Any double complex K has an associated "single" complex (also denoted by K) that is supplied with the "diagonal" gradation $K = \Sigma K_n$, with $K_n = \Sigma K_{p,q}$ $(p + q = n)$ and the endomorphism $d = d_1 + d_2\colon K_n \to K_{n+r}$. Note that $d \cdot d = 0$. We shall filter this complex in two ways, denoting the resulting filtered complexes by $'K$ and $''K$, respectively. We set

$$'K^{(p)} = \sum_{j \leq p} K_{i,j} \quad \text{and} \quad ''K^{(p)} = \sum_{i \leq p} K_{i,j}\,.$$

(For cochain complexes we use the decreasing filtrations $'K_{(p)} = \sum_{i \leq p} K^{i,j}$ and $''K_{(p)} = \sum_{i \leq p} K^{i,j}$.)

With every graded filtered module K with a "differential" endomorphism d such that $d \cdot d = 0$ there is associated a sequence of bigraded differential modules

$$E^r = \left\{ \sum_{p,q} E^r_{p,q},\ d^r \right\}$$

such that d^r is homogeneous of bidegree $(-r, r - 1)$ and such that E^{r+1} is isomorphic to $H(E^r)$ as a bigraded module. Such sequences are called *spectral sequences*, and their properties are nicely developed in Chapter 4 of [2]. When (as is the case here) K is properly bigraded, the sequence $E^r_{p,q}$ is stationary (that is,

$E^r_{p,q} \cong E^{r+1}_{p,q}$) for each fixed p, q, for $r \geq N(p, q)$, so that the spectral sequence "converges" to a bigraded limit module E^∞. A homomorphism $f: K \to L$ of graded, filtered, differential modules, that preserves gradation and filtration, and such that $fd(x) = df(x)$ for all x, is called *admissible*. Admissible homomorphisms induce homomorphisms $f^{(r)}: E^r(K) \to E^r(L)$, of bidegree (0, 0), that preserve identities and compositions; therefore we can consider a spectral sequence as a sequence of functors on the category of differential filtered graded modules. The structure of the E^2- and E^∞- terms of the sequence is known : in particular, for the spectral sequences $'E^r$ and $''E^r$ arising from $'K$ and $''K$, we have

$$'E^2_{q,p} \cong {}^1H_{p,q}({}^2H(K)) \quad \text{and} \quad ''E^2_{p,q} \cong {}^2H_{q,p}({}^1H(K)),$$

$$H_n(K) = \sum_{p+q=n} {}'E^\infty_{p,q} = \sum_{p+q=n} {}''E^\infty_{p,q}.$$

Assume now that we are dealing only with regular double chain complexes, and observe that $'E^2_{p,q} = {}''E^2_{p,q} = 0$ except for $q = 0$, whence $'E^2_{p,q} = {}'E^\infty_{p,q}$ and $''E^2_{p,q} = {}''E^\infty_{p,q}$. Thus

$$H_n(K) \cong {}^1H_{n,o}({}^2H(K)) = {}'E^\infty_{n,o} \cong {}^2H_{o,n}({}^1H(K)) = {}''E^\infty_{n,o}.$$

The isomorphisms are "natural" in the following sense.

Given any regular double chain complex K, we can form regular quotient double complexes as follows. Let

$$X^K_{p,q} = K_{p,q}/K_{p,q} = 0 \text{ if } q > 0, \qquad Y^K_{p,q} = K_{p,q}/K_{p,q} = 0 \text{ if } p > 0,$$

$$X^K_{p,q} = K_{p,o}/{}^2B_{p,o} = {}^2H_{p,o}(K), \qquad Y^K_{o,q} = K_{o,q}/{}^1B_{o,q} = {}^1H_{o,q}(K).$$

The important property of these subcomplexes is that they are their own associated "single" complexes, that is, $X_n = X_{n,o}$ with $d = d_1$, and $Y_n = Y_{o,n}$ with $d = d_2$. The projections onto the quotient double complexes induce chain maps

$$X^K \leftarrow K \rightarrow Y^K;$$

these induce isomorphisms

$${}^1H_{n,o}(X^K) = H_n(X^K) \xleftarrow{\cong} H_n(K) \xrightarrow{\cong} H_n(Y^K) = {}^2H_{o,n}(Y^K),$$

in view of the fact that the induced maps of the spectral sequence are isomorphisms on the E^2-term.

Given any map $f: K \to L$ of regular double complexes, we obtain a commutative diagram of the associated chain complexes

$$\begin{array}{ccccc} X^K & \longleftarrow & K & \longrightarrow & Y^K \\ \downarrow f^X & & \downarrow f & & \downarrow f^Y \\ X^L & \longleftarrow & L & \longrightarrow & Y^L \end{array}$$

This in turn yields the commutative diagram

$$\begin{array}{ccccc} H_n(X^K) & \xleftarrow{\approx} & H_n(K) & \xrightarrow{\approx} & H_n(Y^K) \\ \downarrow f_*^X & & \downarrow f_* & & \downarrow f_*^Y \\ H_n(X^L) & \xleftarrow[\approx]{} & H_n(L) & \xrightarrow[\approx]{} & H_n(Y^L) \end{array} .$$

Further, any short exact sequence of regular double complexes

$$0 \to K \to L \to M \to 0$$

induces a commutative diagram of chain complexes

$$\begin{array}{ccccccccc} 0 & \to & X^K & \to & X^L & \to & X^M & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & K & \to & L & \to & M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Y^K & \to & Y^L & \to & Y^M & \to & 0 \end{array}$$

with exact rows, whence on the homology level we have the commutative diagram

$$\begin{array}{ccccc} H(X^M) & \xleftarrow{\approx} & H(M) & \xrightarrow{\approx} & H(Y^M) \\ \partial_* \downarrow & & \partial_* \downarrow & & \partial_* \downarrow \\ H(X^K) & \xleftarrow[\approx]{} & H(K) & \xrightarrow[\approx]{} & H(Y^K) \end{array} .$$

The preceding remarks apply almost without change to regular double cochain complexes. The basic difference is that if $K^{p,q}$ is a regular cochain complex, we define

$$X_K^{p,q} = 0 \text{ for } p > 0, \qquad Y_K^{p,q} = 0 \text{ for } p > 0,$$

$$X_K^{p,o} = {}^2Z^{p,o} = {}^2H^{p,o}(K), \qquad Y_K^{o,q} = {}^1Z^{o,p} = {}^1H^{o,q}(K) .$$

In this case X_K and Y_K are subcomplexes instead of quotient complexes, and we have injections

$$X_K \to K \leftarrow Y_K$$

that induce isomorphisms

$$H^n(X_K) \xrightarrow{\approx} H^n(K) \xleftarrow{\approx} H^n(Y_K) .$$

Thus we have shown

LEMMA 1. *The three functors on the category of regular double chain* {*cochain*} *complexes that assign to* K *the graded modules* $H_*(X^K)$, $H_*(K)$, *and* $H_*(Y^K)$ {$H^*(X_K)$, $H^*(K)$, *and* $H^*(Y_K)$}, *respectively, are naturally isomorphic.*

2. THE CATEGORY OF RELATIONS

Let $\mathscr{S}$ denote the category whose objects are pairs (S_1, S_2) consisting of a set and a subset (possibly empty), and whose maps are functions $f: (S_1, S_2) \to (T_1, T_2)$ mapping S_1 into T_1 and S_2 into T_2. The *cartesian product* of two pairs is the pair $(S_1 \times T_1, S_2 \times T_2)$, and a *relation* between (S_1, S_2) and (T_1, T_2) is a subpair of the cartesian product, that is, a pair (R_1, R_2) such that $R_1 \subset S_1 \times T_1$ and $R_2 \subset S_2 \times T_2$. If $f: (S_1, S_2) \to (S_1', S_2')$ and $g: (T_1, T_2) \to (T_1', T_2')$, we have a function

$$f \times g: (S_1 \times T_1, S_2 \times T_2) \to (S_1' \times T_1', S_2' \times T_2').$$

If

$$(R_1', R_2') \subset (S_1' \times T_1', S_2' \times T_2') \quad \text{and} \quad f \times g(R_1, R_2) \subset (R_1', R_2'),$$

then the restriction of $f \times g$ to (R_1, R_2) is called a *relation map*. Note that not all functions from (R_1, R_2) to (R_1', R_2') are relation maps, only those induced by restricting maps of $(S_1 \times T_1, S_2 \times T_2)$ into $(S_1' \times T_1', S_2' \times T_2')$. We shall emphasize this fact by using the notation $f \times g: (R_1, R_2) \to (R_1', R_2')$. The collection of relations and relation maps is a subcategory of $\mathscr{S}$, which we denote by $\mathscr{R}$.

With every relation (R_1, R_2) are associated two abstract simplicial pairs $(\tilde{S}_1, \tilde{S}_2)$ and $(\tilde{T}_1, \tilde{T}_2)$. A p-simplex σ^p of $\tilde{S}_i$ is a set of $p + 1$ points $(s_0, \cdots, s_p)$ in S_1 that are all R_1-related to a common $t \in T_1$, in other words, such that the points $(s_0, t), \cdots, (s_p, t)$ are all in R_1. The subcomplex $\tilde{S}_2$ is defined by changing the subscript 1 to 2 in the preceding sentence, and $\tilde{T}_1$ and $\tilde{T}_2$ are defined similarly: a *q-simplex* of $\tilde{T}_i$ is a set of $q + 1$ points of T_i such that, for some $s \in S_i$, the points $(s, t_0), \cdots, (s, t_q)$ are in R_i ($i = 1$ or $i = 2$). Observe that any relation map $f \times g: (R_1, R_2) \to (R_1', R_2')$ induces a pair of simplicial maps

$$\tilde{f}: (\tilde{S}_1, \tilde{S}_2) \to (\tilde{S}_1', \tilde{S}_2') \quad \text{and} \quad \tilde{g}: (\tilde{T}_1, \tilde{T}_2) \to (\tilde{T}_1', \tilde{T}_2').$$

If K is any simplicial complex, an ordered *n-simplex* of K is an ordered n-tuple $\langle v_0, \cdots, v_n \rangle$ of vertices (not necessarily distinct) of K, all of which lie in a common simplex of K. An ordered p-simplex $\sigma^p = \langle s_0, \cdots, s_p \rangle$ of $\tilde{S}_i$ and an ordered q-simplex $\tau^q = \langle t_0, \cdots, t_q \rangle$ of $\tilde{T}_i$ are said to be R_i-*related* ($i = 1$ or $i = 2$) if $(s_j, t_k) \in R_i$ for all $j = 0, 1, \cdots, p$ and $k = 0, 1, \cdots, q$. The set of pairs (σ^p, τ^q) that are R_i-related is denoted by $\Omega^i_{p,q}$. Note that $\Omega^2_{p,q}$ is a subset of $\Omega^1_{p,q}$.

Let $C(R_i)$ be the submodule of $C_p(\tilde{S}_i) \otimes C_q(\tilde{T}_i)$ generated by $\sigma^p \otimes \tau^q$ for all $(\sigma^p, \tau^q) \in \Omega^i_{p,q}$. We define the endomorphisms d_1 and d_2 as follows:

$$d_1(\sigma^p \otimes \tau^q) = \sum_0^p (-1)^i \partial^i \sigma^p \otimes \tau^q,$$

$$d_2(\sigma^p \otimes \tau^q) = \sum_0^q (-1)^{i+q} \sigma^p \otimes \partial^i \tau^q,$$

where $\partial^i \langle V_0, \cdots, V_q \rangle = \langle V_0, \cdots, \hat{V}_i, \cdots, V_q \rangle$ is the ordered simplex with its i-th entry deleted. This is legitimate, since if $\sigma^p \otimes \tau^q$ is a related pair, then so are $\partial^i \sigma^p \otimes \tau^q$ and $\sigma^p \otimes \partial^i \tau^q$ for all i. Since $d_1^2 = d_2^2 = d_1 d_2 + d_2 d_1 = 0$, $C(R_i)$ is a double chain complex. Observe that $C(R_2)$ is a subcomplex of $C(R_1)$, and define $C(R_1, R_2)$ to be the quotient double complex $C(R_1)/C(R_2)$. The simplical maps $\tilde{f}$, $\tilde{g}$ induced by a relation map $f \times g$: $(R_1, R_2) \to (\bar{R}_1, \bar{R}_2)$ induce a function from $(\Omega^1_{p,q}\Omega^2_{p,q}) \to (\bar{\Omega}^1_{p,q}\bar{\Omega}^2_{p,q})$ that defines a map of double complexes

$$(f \times g)_{\#}: (C(R_1), C(R_2)) \to (C(\bar{R}_1), C(\bar{R}_2)).$$

Let G be an arbitrary A-module. We define

$$C_{p,q}(R_1, R_2; G) = C_{p,q}(R_1, R_2) \otimes_A G,$$

$$C^{p,q}(R_1, R_2; G) = \mathrm{Hom}_A (C_{p,q}(R_1, R_2), G)$$

and call these things the *double complex of ordered chains* (or cochains) *with coefficients in* G, of the relation pair (R_1, R_2). Finally, we define cohomology and homology modules with coefficients in G to be those of the associated "diagonal" *complex* $H(R_1, R_2; G) = H(C(R_1, R_2; G))$, the homomorphisms being induced in the usual fashion.

THEOREM 2.1. *The correspondence we have just described, which associates with* (R_1, R_2) *the chain (or cochain) complex* $C(R_1, R_2; G)$ *and with the relation map* $f \times g$ *the map* $(f \times g)_{\#}$, *is a covariant (contravariant) functor from* $\mathscr{R}$ *to the category* $\mathscr{N}$ *of regular double chain (or cochain) complexes.*

Proof. The only statement which is not clear is that $C(R_1, R_2; G)$ is regular, and to prove this it suffices to show that $C(R_1, R_2)$ is regular, since

$$0 \otimes_A G = \mathrm{Hom}_A (0, G) = 0.$$

For i = 1, 2, let $R^q_i[\sigma^p]$ be the set of τ^q related to σ^p, and $R^p_i[\tau^q]$ the set of σ^p related to τ^q. Observe that

(1) the three statements $\tau^q \in R^q_i[\sigma^p]$, $\sigma^p \in R^p_i[\tau^q]$, and $(\sigma^p, \tau^q) \in \Omega^i_{p,q}$ are equivalent,

(2) $R_i[\sigma^p] = \bigcup_q R^q_i[\sigma^p]$ and $R_i[\tau^q]$ are subcomplexes.

Since any chain in $C_{p,q}(R_1)$ can be written in either of the forms $\Sigma\, \sigma^p_i \otimes A^q_i$ and $\Sigma\, B^p_i \otimes \tau^q_i$, by grouping the terms containing a particular σ^p or τ^q, we see that

$$C_{p,q}(R_1) \cong \sum_{\sigma^p} C_q(R_1[\sigma^p]) \cong \sum_{\tau^q} C_p(R_1[\tau^q]),$$

whence

$${}^2H_{p,q}(R_1) = \sum_{\sigma^p} H_q(R_1[\sigma^p]) \quad \text{and} \quad {}^1H_{p,q}(R_1) = \sum_{\tau^q} H_p(R_1[\tau^q]).$$

Now $R_1[\sigma^p]$ and $R_1[\tau^q]$ are subcomplexes spanned by a set of vertices every finite subset of which is a simplex; hence they are cone complexes, and therefore they are homologically trivial. Thus

$${}^2H_{p,q}(R_1) = 0 \text{ for } q > 0 \quad \text{and} \quad {}^1H_{p,q}(R_1) = 0 \text{ for } p > 0;$$

that is, $C_{p,q}(R_1)$ is regular. The same argument shows that $C_{p,q}(R_2)$ is regular, and $C_{p,q}(R_1, R_2)$ is regular since, if K and L are cone complexes ($L \subset K$), then $H(K, L) = 0$ for $m > 0$.

We have seen that a relation pair (R_1, R_2) is associated with two simplical pairs $(\tilde{S}_1, \tilde{S}_2)$ and $(\tilde{T}_1, \tilde{T}_2)$. Let $H(S_1, S_2)$ and $H(T_1, T_2)$ denote the usual homology or cohomology theories for abstract simplicial complexes.

THEOREM 2.2. *There exist natural transformations*

$$H_*(S_1, S_2; G) \overset{P_*}{\leftarrow} H_*(R_1, R_2; G) \overset{G_*}{\rightarrow} H_*(T_1, T_2; G),$$

$$H_*(S_1, S_2; G) \overset{I^*}{\rightarrow} H_*(R_1, R_2; G) \overset{J^*}{\leftarrow} H_*(T_1, T_2; G)$$

that are isomorphisms of these homology and cohomology theories.

Proof. The proof consists in identifying X^K and Y^K when $K = C(R_1, R_2)$ and applying Lemma 1. This gives

$$X_n^K = {}^2H_{n,0}(K) \cong \sum_{\sigma^n} \sigma^n \otimes H_0(R_1[\sigma^n], R_2[\sigma^n]).$$

$H_0(R_1[\sigma^n], R_2[\sigma^n])$ is either free on one generator (denoted by $g(\sigma^n)$), in case $R_2[\sigma^n]$ is empty, or it is 0, in case $R_2[\sigma^n]$ is not empty. Since $\sigma^n \in \tilde{S}_2$ if and only if $R_2[\sigma^n]$ is not empty, the correspondence $\sigma^n \rightarrow \sigma^n \otimes g(\sigma^n)$ defines a natural isomorphism $C_n(S_1, S_2) \rightarrow X_n^K$. Clearly, the boundary operators correspond, so that $H_n(X^K) \cong H_n(\tilde{S}_1, \tilde{S}_2)$. The "dual" argument shows that $H_n(Y^K)$ is naturally isomorphic to $H_n(\tilde{T}_1, \tilde{T}_2)$.

REFERENCES

1. C. H. Dowker, *Homology groups of relations*, Ann. of Math. (2) 56 (1952), 84-95.
2. R. Godement, *Topologie algébrique et théorie des faisceaux*, Actualités Sci. Ind. no. 1252, Hermann, Paris, 1958.

New Mexico State University.