EXTREMAL LENGTH DEFINITIONS FOR THE CONFORMAL CAPACITY OF RINGS IN SPACE

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1. INTRODUCTION

It is well known that many important properties of conformal and quasiconformal mappings in the plane can be deduced from studying what happens to the modulus or, equivalently, to the capacity of rings. Recently Loewner showed in [7] how this method can be extended to higher dimensions by defining a conformal capacity for rings in Euclidean 3-space by means of a Dirichlet integral. The present author has found Loewner's idea very fruitful, and he has used it to establish a number of results on conformal and quasiconformal mappings in space. (See [2], [3] and [4].)

J. Väisälä, in [13] and [14], and B. V. Šabat, in [8] and [9], have investigated quasiconformal mappings in space, using extremal lengths. In doing so, each of them has tacitly introduced a new kind of conformal capacity for a space ring R. Väisälä considered the module of the family of curves in R that join the boundary components of R, while Šabat studied the module of a family of surfaces that separate the boundary components of R.

In the first part of this paper we shall give two extremal length definitions for the conformal capacity of a ring in space. The first of these is essentially equivalent to Väisälä's definition, while the second is a slightly modified version of Šabat's definition. We shall then show that these two extremal length definitions are equivalent to the Dirichlet integral definition due to Loewner.

In the second part of the paper, we use these extremal length definitions to obtain upper and lower bounds for the moduli of some rings in space that have axial symmetry. In particular, we show that the moduli of the Grötzsch and Teichmüller extremal rings in space are greater than or equal to the moduli of the corresponding rings in the plane.

2. NOTATION

We consider sets in the Möbius space, that is, in the finite Euclidean 3-space plus the point at infinity. Points will be designated by capital letters P and Q or by small letters x and y. In the latter case, x_1, x_2, x_3 will denote the coordinates for x, and similarly for y; x will denote the point at infinity if any one of its coordinates x_i is infinite. Points are treated as vectors, and $|P|$ and $|x|$ will denote the norms of P and x, respectively.

Given a point P and sets E and F, we let $\rho(P, E)$ denote the distance between P and E, and $\rho(E, F)$ the distance between E and F. We further let ∂E, $\mathscr{C}E$, and $\overline{E}$ denote the boundary, complement, and closure of E, respectively.

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Finally, for each function $u = u(x)$, ∇u will denote the vector $\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right)$, defined at each point where the three partial derivatives exist.

3. CONFORMAL CAPACITY OF A RING

A *ring* R is defined as a finite domain whose complement in the Möbius space consists of two components, C_0 and C_1. We let C_1 denote the component that contains the point at infinity, and we set $B_0 = \partial C_0$, $B_1 = \partial C_1$. Then B_0 and B_1 are simply the components of ∂R.

Next, following Loewner, we define the *conformal capacity* of a ring R as

$$\Gamma(R) = \inf_u \int_R |\nabla u|^3 \, d\omega, \tag{1}$$

where u ranges over all functions that are continuously differentiable in R and have boundary values 0 on B_0 and 1 on B_1. When R is unbounded, this last requirement will mean that $u(x) \to 1$ as $|x| \to \infty$ in R. Loewner proved, in [7], that $\Gamma(R) > 0$ if and only if R has nondegenerate boundary components, that is, if neither B_0 nor B_1 reduces to a point.

We can enlarge the class of competing functions u as follows. A function u is said to be *absolutely continuous on lines*, or simply ACL, in a finite domain D if, for each sphere U with $\overline{U} \subset D$, u is absolutely continuous on almost all line segments in U parallel to the coordinate axes. If u is continuous and ACL in a ring R, then u has partial derivatives a. e. in R. If, in addition, u has boundary values 0 on B_0 and 1 on B_1, then

$$\Gamma(R) \leq \int_R |\nabla u|^3 \, d\omega .$$

(See Lemma 1 of [3].) We call such a function u an *admissible function* for R, and we see that we may take the infimum in (1) over this class of functions without affecting the value of $\Gamma(R)$.

If R has nondegenerate boundary components, there exists a unique admissible function u for which

$$\Gamma(R) = \int_R |\nabla u|^3 \, d\omega .$$

(See Theorem 1 of [4].) We call u the *extremal function* for R. It satisfies the variational condition

$$\int_R |\nabla u| \, \nabla u \cdot \nabla w \, d\omega = 0, \tag{2}$$

where w is any function that is continuous and ACL in R, that has boundary value 0 on ∂R, and for which $|\nabla w|$ is L^3-integrable. (See Corollary 2 of [4].)

4. EXTREMAL LENGTHS

Let R be a ring. A curve γ is said to *join the boundary components in* R if γ lies in R, except for its endpoints, and if one of these endpoints lies in B_0 and the other in B_1. A compact set Σ is said to *separate the boundary components of* R if $\Sigma \subset R$ and if C_0 and C_1 lie in different components of $\mathscr{C}\Sigma$. Now, given a function f that is nonnegative and Borel measurable in R, we let

$$L(f) = L(R, f) = \inf_{\gamma} \int_{\gamma} f\, ds,$$

where the infimum is taken over all locally rectifiable curves γ that join the boundary components in R. Next we let

$$A(f) = A(R, f) = \inf_{\Sigma} \int_{\Sigma} f^2\, d\sigma,$$

where now the infimum is taken over all compact piecewise smooth surfaces Σ that separate the boundary components of R. Finally we set

$$V(f) = V(R, f) = \int_{R} f^3\, d\omega.$$

The main purpose of this paper is to establish the following two extremal length definitions for the conformal capacity of a ring in space.

THEOREM 1. *If* R *is a ring, then*

$$\inf_f \frac{V(f)}{L(f)^3} = \Gamma(R) = \sup_f \frac{A(f)^3}{V(f)^2}. \tag{3}$$

The infimum is taken over all f *for which* V(f) *and* L(f) *are not simultaneously* 0 *or* ∞, *and the supremum over all* f *for which* A(f) *and* V(f) *are not simultaneously* 0 *or* ∞.

J. Väisälä informs me that he also has proved a result similar to the first part of (3).

5. PRELIMINARY RESULTS

We list three lemmas that will be used in the proof of Theorem 1.

LEMMA 1. *Let* S *be a spherical surface of radius* r, *and let* f *be nonnegative and Borel measurable on* S. *Then each pair of points on* S *can be joined by a circular arc* $\alpha \subset S$ *such that*

$$\left(\int_{\alpha} f\, ds \right)^3 \leq Ar \int_{S} f^3\, d\sigma,$$

where A *is an absolute constant.*

This is established by replacing $|\nabla u|$ by f in the proof of Lemma 1 in [4].

LEMMA 2. *Let* R *be a ring with nondegenerate boundary components, and let* f *be nonnegative, Borel measurable, and* L^3*-integrable in the finite space. For each* $a > 0$ *there exists a* $b > 0$ *with the following property. If* P_0 *and* P_1 *are the end-points of a rectifiable curve* β, *if* $\rho(P_0, B_0) < b$, *and if either* $\rho(P_1, B_1) < b$ *or* $|P_1| > 4/b$, *then*

$$\int_\beta f\,ds \geq L(f) - a. \tag{4}$$

Proof. Fix $a > 0$ and choose $c > 0$ so that

$$\left(\frac{a}{2}\right)^3 = \frac{Ac}{\log 2}, \tag{5}$$

where A is the constant of Lemma 1. Then we can find a number b $(0 < b < 1/\sqrt{2})$ such that the following is true. C_0 lies in the sphere $|x| < 1/b$ and has diameter greater than 4b, C_1 meets the surface $|x| = 2/b$, $3b < \rho(B_0, B_1)$,

$$\int_{|x-Q|<2b} f^3\,d\omega \leq c \tag{6}$$

for every point Q, and

$$\int_{|x|>2/b} f^3\,d\omega \leq c. \tag{7}$$

Now let β be a rectifiable curve joining P_0 and P_1, where $\rho(P_0, B_0) < b$ and where either $\rho(P_1, B_1) < b$ or $|P_1| > 4/b$. In order to establish (4), it suffices to show for $i = 0, 1$ that either β meets B_i or there exists a circular arc α_i joining β to B_i such that

$$\int_{\alpha_i} f\,ds \leq \frac{a}{2}.$$

Then, for example, if β does not meet ∂R, $\beta \cup \alpha_0 \cup \alpha_1$ will contain a rectifiable curve γ that joins the boundary components in R, and

$$\int_\beta f\,ds \geq \int_\gamma f\,ds - \int_{\alpha_0} f\,ds - \int_{\alpha_1} f\,ds \geq L(f) - a,$$

as desired.

Suppose that $\beta \cap B_0$ is empty, and choose Q_0 in B_0 so that $|P_0 - Q_0| \leq b$. Then (6) implies that there exists a spherical surface S_0, with center at Q_0 and radius r_0 $(b < r_0 < 2b)$, such that

$$r_0 \int_{S_0} f^3\,d\sigma \leq \frac{c}{\log 2}.$$

Now P_0 lies inside, and P_1 outside of S_0. Furthermore, the diameter of S_0 is less than that of B_0. Hence S_0 meets β and B_0, and by Lemma 1 and (5) we can find a circular arc $\alpha_0 \subset S_0$ that joins β to B_0 and for which

$$\int_{\alpha_0} f\,ds \leq \frac{a}{2}.$$

Now suppose that $\beta \cap B_1$ is empty. Then we can find a spherical surface S_1, with center at Q_1 and radius r_1, such that

$$r_1 \int_{S_1} f^3\,d\sigma \leq \frac{c}{\log 2}. \tag{8}$$

When $\rho(P_1, B_1) < b$, we choose Q_1 in B_1 so that $|P_1 - Q_1| \leq b$, and $b < r_1 < 2b$ so that (8) holds. When $|P_1| > 4/b$, we take Q_1 as the origin and, on the basis of (7), choose $2/b < r_1 < 4/b$ so that (8) is valid. In each case it is easy to see that P_0 and P_1 are separated by S_1, and hence that S_1 meets β. The hypotheses further imply, in each case, that $\beta \subset \mathscr{C}C_1$ and that $S_1 \cap C_1$ is not empty. Thus S_1 also meets B_1. Lemma 1 now yields a circular arc $\alpha_1 \subset S_1$, joining β to B_1, such that

$$\int_{\alpha_1} f\,ds \leq \frac{a}{2},$$

and the proof of Lemma 2 is complete.

Given a set Σ and a number $b > 0$, we let $\Sigma(b)$ denote the set of points x for which $\rho(x, \Sigma) < b$.

LEMMA 3. *Let* R *be a ring with nondegenerate boundary components, let* u *be the extremal function for* R, *and let* Σ *be a compact set that separates the boundary components of* R. *Then*

$$\int_{\Sigma(b)} |\nabla u|^2\,d\omega \geq 2b\,\Gamma(R)$$

for $0 < b < \rho(\Sigma, \partial R)$.

Proof. Fix $0 < b < \rho(\Sigma, \partial R)$. Next, for $i = 0, 1$, let D_i be the component of $\mathscr{C}\Sigma$ that contains C_i, and let E_i be the set of points for which $0 < \rho(x, \mathscr{C}D_i) < b$. Then $E_i \subset D_i$ and $E_0 \cup E_1 \subset \Sigma(b)$. Hence it is sufficient to show that

$$\int_{E_i} |\nabla u|^2\,d\omega \geq b\,\Gamma(R) \tag{9}$$

for $i = 0, 1$.

We consider the case where $i = 1$. For this, set $w = v - bu$, where the function v is defined as follows:

$$v(x) = \min(b, \rho(x, \mathscr{C}D_1)).$$

Then w is clearly continuous and ACL in R. Next, it is easy to see that v has boundary values 0 on B_0 and b on B_1, and hence that w has boundary value 0 on ∂R. Finally, $|\nabla v| \leq 1$ a.e. in E_1, while $|\nabla v| = 0$ a.e. in $R - E_1$. This last statement follows from the fact that almost every point of $\mathscr{C}E_1$ is a point of linear density for $\mathscr{C}E_1$ in the directions of the coordinate axes. (See, for example, [10, p. 298].) Since E_1 is bounded, $|\nabla w|$ is L^3-integrable in R, and we can apply the variational condition in (2) to conclude that

$$\int_{E_1} |\nabla u|^2 \, d\omega \geq \int_R |\nabla u| \nabla u \cdot \nabla v \, d\omega = b \int_R |\nabla u|^3 \, d\omega = b\Gamma(R),$$

as desired.

A trivial modification of the argument above yields (9) for $i = 0$, thus completing the proof of Lemma 3.

6. PROOF OF THE FIRST HALF OF THEOREM 1

We prove here that

$$\Gamma(R) = \inf_f \frac{V(f)}{L(f)^3}, \tag{10}$$

where the infimum is taken over all nonnegative Borel measurable functions f with $V(f)$ and $L(f)$ not simultaneously 0 or ∞.

Fix $a > 0$, let u be a continuously differentiable admissible function for R with

$$\int_R |\nabla u|^3 \, d\omega < \Gamma(R) + a,$$

and set $f = |\nabla u|$. Then

$$\int_\gamma f \, ds = \int_\gamma |\nabla u| \, ds \geq 1$$

for each locally rectifiable curve γ that joins the boundary components in R. Hence

$$L(f) \geq 1, \qquad V(f) < \Gamma(R) + a,$$

and letting $a \to 0$, we obtain the inequality

$$\inf_f \frac{V(f)}{L(f)^3} \leq \Gamma(R).$$

To complete the proof for (10), we must show that

$$\Gamma(R) \leq \frac{V(f)}{L(f)^3} \tag{11}$$

for all nonnegative Borel measurable functions f with $V(f)$ and $L(f)$ not simultaneously 0 or ∞. Now (11) is trivial in case $\Gamma(R) = 0$, $L(f) = 0$, or $V(f) = \infty$. Hence we may assume without loss of generality that R has nondegenerate boundary components and that $L(f) \geq 1$ and $V(f) < \infty$.

Let $0 < a < 1$, and extend f to be equal to 0 in $\mathscr{C}R$. Then f satisfies the hypotheses of Lemma 2, and we can find a number b $(0 < b < 1)$ for which the conclusions of Lemma 2 hold. Next, let

$$g(x) = \frac{1}{m(U)} \int_U f(x + y)\, d\omega,$$

where U is the sphere $|y| < b$. Then g is bounded and continuous in the finite space. Moreover,

$$\int_\beta g\, ds \geq L(f) - a \tag{12}$$

for each polygonal arc β joining P_0 and P_1, where P_0 is in B_0 and where either P_1 is in B_1 or $|P_1| \geq 5/b$. For by Fubini's theorem,

$$\int_\beta g(x)\, ds = \int_\beta \left(\frac{1}{m(U)} \int_U f(x + y)\, d\omega(y)\right) ds(x) = \frac{1}{m(U)} \int_U \left(\int_{\beta_y} f(x)\, ds(x)\right) d\omega(y),$$

where β_y denotes the translation of β through the vector y. Lemma 2 then implies that

$$\int_{\beta_y} f\, ds \geq L(f) - a$$

for all $|y| < b$, and hence (12) follows.

Since g is bounded and since B_0 and B_1 are nondegenerate, (12) implies that $L(f) < \infty$. Now for each x let

$$u(x) = \inf_\beta \int_\beta g\, ds,$$

where β is any polygonal arc joining x to B_0, and set

$$v(x) = \min\left(1, \frac{u(x)}{L(f) - a}\right).$$

It is easy to see that v satisfies a uniform Lipschitz condition, that $v = 0$ on B_0, and that $v = 1$ on B_1 and outside the sphere $|x| \leq 5/b$. Hence v is admissible for R, and since

$$|\nabla v| \leq \frac{g}{L(f) - a}$$

a.e. in R, we obtain the inequality

$$\Gamma(R) \leq \frac{1}{(L(f) - a)^3} \int_R g^3\, d\omega.$$

Minkowski's inequality (see [5, p. 148]) implies that

$$\int_R g(x)^3\, d\omega \leq \left(\frac{1}{m(U)} \int_U \left[\int_R f(x + y)^3\, d\omega(x)\right]^{1/3} d\omega(y)\right)^3 \leq V(f).$$

Thus

$$\Gamma(R) \leq \frac{V(f)}{(L(f) - a)^3},$$

and, letting $a \to 0$, we obtain (11). This completes the proof of the first half of Theorem 1.

7. PROOF OF THE SECOND HALF OF THEOREM 1

We prove next that

$$\Gamma(R) = \sup_f \frac{A(f)^3}{V(f)^2}, \tag{13}$$

where the supremum is taken over all nonnegative Borel measurable functions f with A(f) and V(f) not simultaneously 0 or ∞.

Fix $a > 0$, and let f be nonnegative and Borel measurable in R. By the result proved in Section 7 of [3], we can find an admissible function u with the following properties: u is piecewise linear in R,

$$\int_R |\nabla u|^3 \, d\omega < \Gamma(R) + a,$$

and, for all but a finite set of b in $0 < b < 1$, the points where $u = b$ form a polyhedral surface Σ that separates the boundary components of R. Thus

$$A(f) \leq \int_\Sigma f^2 \, d\sigma,$$

and, integrating over all such b, we find that

$$A(f) \leq \int_0^1 \left(\int_\Sigma f^2 \, d\sigma \right) db \leq \int_R |\nabla u| \, d\omega.$$

Hölder's inequality yields the result

$$A(f)^3 \leq \left(\int_R f^3 \, d\omega \right)^2 \left(\int_R |\nabla u|^3 \, d\omega \right) < V(f)^2 \, (\Gamma(R) + a),$$

and, letting $a \to 0$, we conclude that

$$\frac{A(f)^3}{V(f)^2} \leq \Gamma(R)$$

for all f with A(f) and V(f) not simultaneously 0 or ∞.

To complete the proof of (13), we must show that

$$\Gamma(R) \leq \sup_f \frac{A(f)^3}{V(f)^2}. \tag{14}$$

This is clearly so when $\Gamma(R) = 0$, and hence we need only consider the case where R has nondegenerate boundary components.

Let u be the extremal function for R, and, for each $r > 0$, let

$$f(x, r) = \left(\frac{1}{m(U)} \int_U |\nabla u(x+y)|^2 \, d\omega\right)^{1/2},$$

where U is the sphere $|y| < r$ and where $|\nabla u|$ is taken as 0 in $\mathscr{C}R$. Next, for each $a > 0$, let

$$g(x) = \sup_{0<r<a} f(x, r), \qquad h(x) = \sup_{0<r<\infty} f(x, r).$$

Both g and h are nonnegative Borel measurable functions, and a form of the Hardy-Littlewood maximal theorem, due to K. T. Smith, implies that h is L^3-integrable over R. (See Theorem 1 of [11].) Then, since

$$\lim_{r \to 0} f(x, r) = |\nabla u(x)|$$

a.e. in R, we conclude from Lebesgue's dominated convergence theorem that

$$\lim_{a \to 0} V(g) = \int_R |\nabla u|^3 \, d\omega = \Gamma(R). \tag{15}$$

Now fix $a > 0$, let Σ be a compact piecewise smooth surface that separates the boundary components of R, and choose $b > 0$ and $r > 0$ so that $r < a$ and $b + r < \rho(\Sigma, \partial R)$. By Fubini's theorem,

$$\int_{\Sigma(b)} f(x, r)^2 \, d\omega = \int_{\Sigma(b)} \left(\frac{1}{m(U)} \int_U |\nabla u(x+y)|^2 \, d\omega(y)\right) d\omega(x)$$

$$= \frac{1}{m(U)} \int_U \left(\int_{\Sigma_y(b)} |\nabla u(x)|^2 \, d\omega(x)\right) d\omega(y),$$

where U is the sphere $|y| < r$, Σ_y is the translation of Σ through the vector y, and $\Sigma_y(b)$ is the set of points x with $\rho(x, \Sigma_y) < b$. Lemma 3 implies that

$$\int_{\Sigma_y(b)} |\nabla u|^2 \, d\omega \geq 2b\,\Gamma(R)$$

for all $|y| < r$, and hence that

$$\int_{\Sigma(b)} f(x, r)^2 \, d\omega \geq 2b\,\Gamma(R).$$

Since f is continuous and Σ is piecewise smooth,

$$\int_\Sigma g(x)^2 \, d\sigma \geq \int_\Sigma f(x, r)^2 \, d\sigma = \lim_{b \to 0} \frac{1}{2b} \int_{\Sigma(b)} f(x, r)^2 \, d\omega \geq \Gamma(R),$$

and we conclude that

$$A(g) \geq \Gamma(R).$$

This, together with (15), implies that

$$\Gamma(R) \leq \sup_{0<a<\infty} \frac{A(g)^3}{V(g)^2},$$

and we obtain (14), thus completing the proof of the second half of Theorem 1.

8. MODULUS OF A RING

A plane ring is a finite plane domain whose complement with respect to the extended plane consists of two components. It is well known that each such ring R' can be mapped conformally onto some annulus $a < |z| < b$. The conformal invariant

$$\text{mod } R' = \log \frac{b}{a}$$

is called the modulus of the ring R'.

The situation in space is quite different. First of all, the exterior of the Alexander horned sphere, (see [1] or [6, p. 176]), minus a closed neighborhood of the point at infinity, is a ring according to the definition in Section 3. Hence a space ring may fail to be topologically equivalent to any spherical annulus $a < |x| < b$. Next, even if we restrict ourselves to rings that are homeomorphic to spherical annuli, we see that the only conformal mappings in space are the Möbius transformations. Thus R is conformally equivalent to a spherical annulus if and only if R is bounded by two spherical surfaces or by a spherical surface and a plane.

On the other hand, we can get a satisfactory definition for the modulus of a space ring in terms of its conformal capacity if we set

(16) $$\text{mod } R = \left(\frac{4\pi}{\Gamma(R)}\right)^{1/2}.$$

Then mod R is a conformal invariant, and the modulus of the spherical annulus $a < |x| < b$ turns out to be $\log b/a$. (See Section 2 in [3].)

It is important to obtain estimates for the moduli of certain simple rings, and, applying Theorem 1, we obtain the following result. (Compare Lemma 7 of [3].)

THEOREM 2. *Suppose that* $y(x)$ *is a continuously differentiable homeomorphism of* $a < |x| < b$ *onto a ring* R, *that* $y(x)$ *has a nonvanishing Jacobian, and that* $y(x)$ *maps each radius of* $a < |x| < b$ *onto a curve that is normal to the image of each surface* $|x| = r$. *Then*

(17) $$\int_a^b D_1(r)\frac{dr}{r} \leq \text{mod } R \leq \int_a^b D_2(r)\frac{dr}{r},$$

where, for $a < r < b$,

$$D_1(r) = \min_{|x|=r} \left(\frac{N(x)^3}{J(x)}\right)^{1/2}, \qquad D_2(r) = \max_{|x|=r} \left(\frac{N(x)^3}{J(x)}\right)^{1/2}.$$

Here $J(x)$ *is the absolute value of the Jacobian, and* $N(x)$ *is the stretching normal to* $|x| = r$, *that is*,

$$N(x) = \lim_{h\to 0} \frac{|y(x + hx) - y(x)|}{|hx|} \qquad (h \textit{ real}).$$

Proof. For the first part of (17), let Σ be the image of $|x| = r$ $(a < r < b)$, and let f be nonnegative and Borel measurable in R. Then Σ is a smooth surface that separates the boundary components of R, and $J(x)/N(x)$ is the ratio between corresponding elements of area on Σ and $|x| = r$. Hence

$$A(f) \leq \int_\Sigma f^2\, d\sigma = \int_{|x|=r} f^2 \left(\frac{J}{N}\right) d\sigma,$$

and since $0 < D_1(r) < \infty$, Hölder's inequality implies that

$$A(f)^{3/2} \leq \left(\int_{|x|=r} d\sigma\right)^{1/2} \int_{|x|=r} f^3 \left(\frac{J}{N}\right)^{3/2} d\sigma \leq (4\pi)^{1/2} \frac{r}{D_1(r)} \int_{|x|=r} f^3 J\, d\sigma.$$

If we multiply both sides of this inequality by $\frac{D_1(r)}{r}$ and integrate with respect to r, we obtain the relation

$$A(f)^{3/2} \int_a^b D_1(r) \frac{dr}{r} \leq (4\pi)^{1/2} \int_{a<|x|<b} f^3 J\, d\omega = (4\pi)^{1/2} V(f).$$

This means that

$$\frac{A(f)^3}{V(f)^2} \leq 4\pi \left(\int_a^b D_1(r) \frac{dr}{r}\right)^{-2}$$

whenever A(f) and V(f) are not simultaneously 0 or ∞. Taking the supremum over all such f yields the inequality

$$\Gamma(R) \leq 4\pi \left(\int_a^b D_1(r) \frac{dr}{r}\right)^{-2},$$

and this together with (16) gives the first half of (17).

For the second part of (17), let γ be the image of a fixed radius of $a < |x| < b$, and let f be nonnegative and Borel measurable in R. Then γ is a locally rectifiable curve that joins the boundary components in R, and N(x) is the ratio between corresponding elements of length on γ and the radius of $a < |x| < b$. Thus

$$L(f) \leq \int_\gamma f\, ds = \int_a^b f N\, dr \leq \int_a^b f D_2^{2/3} J^{1/3}\, dr,$$

and Hölder's inequality gives

$$L(f)^3 \le \left(\int_a^b D_2(r)\frac{dr}{r}\right)^2 \int_a^b f^3 J r^2 \, dr .$$

If we integrate over all the radii of $a < |x| < b$, we obtain the relation

$$4\pi L(f)^3 \le \left(\int_a^b D_2(r)\frac{dr}{r}\right)^2 \int_{a<|x|<b} f^3 J \, d\omega = \left(\int_a^b D_2(r)\frac{dr}{r}\right)^2 V(f) .$$

This means that

$$4\pi \left(\int_a^b D_2(r)\frac{dr}{r}\right)^{-2} \le \frac{V(f)}{L(f)^3}$$

whenever $L(f)$ and $V(f)$ are not simultaneously 0 or ∞. Taking the infimum over all such f yields the inequality

$$4\pi \left(\int_a^b D_2(r)\frac{dr}{r}\right)^{-2} \le \Gamma(R) ,$$

and this together with (16) completes the proof of Theorem 2.

9. AN APPLICATION

In conclusion we show how Theorem 2 can be used to estimate the moduli of some simple space rings that have axial symmetry.

Let R' be a plane ring in the $y_1 y_2$-plane, let R' be symmetric in the y_1-axis, and let R be the space ring obtained by revolving R' about the y_1-axis. Then there exists a conformal mapping $y_1 + iy_2 = f(x_1 + ix_2)$, of some plane annulus $a < |x_1 + ix_2| < b$ in the $x_1 x_2$-plane onto R', that preserves symmetries with respect to the x_1- and y_1-axes. If we introduce polar coordinates (s, ϕ) and (t, θ) in the $x_2 x_3$- and $y_2 y_3$-planes, respectively, we obtain a homeomorphism $y(x)$ of the spherical annulus $a < |x| < b$ onto R by letting

$$y_1 + it = f(x_1 + is) \qquad \text{and} \qquad \theta = \phi .$$

It is easy to verify that $y(x)$ satisfies the hypotheses of Theorem 2 and that

$$\frac{N(x)^3}{J(x)} = \left| \frac{s}{t} f'(x_1 + is) \right|$$

for each point x. Hence

$$(18) \quad \begin{cases} D_1(r) = \min\limits_{|x_1+ix_2|=r} \left| \dfrac{x_2}{y_2} f'(x_1 + ix_2) \right|^{1/2} , \\[2ex] D_2(r) = \max\limits_{|x_1+ix_2|=r} \left| \dfrac{x_2}{y_2} f'(x_1 + ix_2) \right|^{1/2} . \end{cases}$$

We consider an example. Fix $a > 0$, and let R' be the plane ring in the $y_1 y_2$-plane bounded by the segment $-1 \leq y_1 \leq 0$, $y_2 = 0$ and by the ray $a \leq y_1 \leq \infty$, $y_2 = 0$. Next, let R be the space ring formed by revolving R' about the y_1-axis, that is, the ring bounded by the segment $-1 \leq y_1 \leq 0$, $y_2 = y_3 = 0$ and by the ray $a \leq y_1 \leq \infty$, $y_2 = y_3 = 0$. R' is the Teichmüller extremal ring [12, pp. 637-639], and R is its counterpart in space.

Now we can find a conformal mapping $y_1 + iy_2 = f(x_1 + ix_2)$, of some plane annulus $1 < |x_1 + ix_2| < b$ onto R', that preserves symmetries in the x_1- and y_1-axes. The inverse mapping $x_1 + ix_2 = g(y_1 + iy_2)$ is analytic in $y_2 > 0$ and sends this half-plane either into $x_2 > 0$ or into $x_2 < 0$. Hence we can apply the half-plane form of Schwarz's Lemma to conclude that

$$|g'(y_1 + iy_2)| \leq \left|\frac{x_2}{y_2}\right|$$

for $y_2 > 0$. By symmetry this holds for all relevant $y_1 + iy_2$. Thus

$$\left|\frac{x_2}{y_2} f'(x_1 + ix_2)\right| \geq 1$$

for $1 < |x_1 + ix_2| < b$, and together with (17) and (18) this yields the inequalities

$$\log b \leq \int_1^b D_1(r)\frac{dr}{r} \leq \operatorname{mod} R. \tag{19}$$

When $a > 1$, we can apply the reflection principle to show that (19) still holds for the case where R' is the plane ring bounded by the circle $|y_1 + iy_2| = 1$ and by the ray $a \leq y_1 \leq \infty$, $y_2 = 0$, and where R is the ring bounded by $|y| = 1$ and by $a \leq y_1 \leq \infty$, $y_2 = y_3 = 0$. In this case, R' is the Grötzsch extremal ring [12, pp. 631-635] and R its analogue in space.

In both of the cases considered above, $\log b$ is the modulus of the plane ring R'. We thus obtain the following result.

COROLLARY. *The modulus of the Grötzsch ring in space is not less than the modulus of the corresponding Grötzsch ring in the plane. Similarly, the modulus of the Teichmüller ring in space is not less than the modulus of the corresponding Teichmüller ring in the plane.*

If we let $\log \Phi(a)$ denote the modulus of the Grötzsch ring bounded by $|x| = 1$ and by $a \leq x_1 \leq \infty$, $x_2 = x_3 = 0$, then it is easy to show that $\Phi(a)/a$ is nondecreasing in $1 < a < \infty$, and hence that

$$\lim_{a \to \infty} \frac{\Phi(a)}{a} = \lambda .$$

The exact value of λ is not yet known. However, combining the Corollary above with known results on the Grötzsch ring in the plane and with Lemma 8 of [3], we conclude that $4 \leq \lambda \leq 12.4 \cdots$.

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