ON ASYMMETRIC DIOPHANTINE APPROXIMATIONS

Ivan Niven

Our purpose is to give a brief proof of the following theorem of B. Segre [5].

Let τ be any non-negative real number. Every irrational number θ has infinitely many rational approximations h/k satisfying

$$(1) \qquad -\frac{1}{(1+4\tau)^{1/2}k^2} < \theta - \frac{h}{k} < \frac{\tau}{(1+4\tau)^{1/2}k^2}.$$

Segre's proof was geometric in nature. C. D. Olds [3] gave a proof using Farey sequences for the cases $\tau > 1$. Proofs by continued fractions have been given by N. Negoescu [2] and R. M. Robinson [4]. W. J. LeVeque [1] showed that (1) holds for at least one of any five consecutive convergents of the continued fraction expansion of θ. We give a short proof of Segre's theorem, using Farey sequences.

LEMMA. *Let θ be an irrational number, and let τ be any nonnegative real number. Let* a/b *and* c/d *be the two consecutive fractions of the Farey series* F_n *between which θ lies, and suppose that*

$$(2) \qquad \frac{a}{b} < \frac{a+c}{b+d} < \theta < \frac{c}{d}.$$

Then (1) *holds with* h/k *replaced by at least one of* a/b, (a + c)/(b + d), *and* c/d.

Proof. Define λ and μ by

$$\lambda = (1+4\tau)^{-1/2} \qquad \text{and} \qquad \mu = \tau(1+4\tau)^{-1/2},$$

so that $\mu = (1-\lambda^2)/4\lambda$ and $0 < \lambda \le 1$. Assuming that the conclusion of the lemma is false, we can write

$$(3) \qquad \theta - \frac{a}{b} \ge \frac{\mu}{b^2}, \quad \theta - \frac{a+c}{b+d} \ge \frac{\mu}{(b+d)^2}, \quad \frac{c}{d} - \theta \ge \frac{\lambda}{d^2}.$$

Adding the first and third of these inequalities, and also the second and third, we obtain the relations

$$\frac{c}{d} - \frac{a}{b} = \frac{1}{bd} \ge \frac{\mu}{b^2} + \frac{\lambda}{d^2},$$

$$\frac{c}{d} - \frac{a+c}{b+d} = \frac{1}{d(b+d)} \ge \frac{\mu}{(b+d)^2} + \frac{\lambda}{d^2},$$

in other words,

$$(4) \qquad \lambda b^2 - bd + \mu d^2 \le 0, \quad \lambda(b+d)^2 - d(b+d) + \mu d^2 \le 0.$$

Received January 2, 1962.
Research supported in part by National Science Foundation Grant G-19016.

Adding the last two inequalities, we conclude

$$(5) \qquad 2\lambda b^2 + (2\lambda - 2)bd + (\lambda + 2\mu - 1)d^2 \leq 0.$$

The discriminant of this quadratic form in b and d is 0, since $\lambda^2 + 4\lambda\mu = 1$. Multiplying (5) by the positive number 2λ, we get

$$\{2\lambda b + (\lambda - 1)d\}^2 \leq 0,$$

so that

$$2\lambda b = (1 - \lambda)d.$$

Thus λ is rational, and equality holds in (5), and hence equality holds in each of the inequalities in (3). The last of these becomes

$$\frac{c}{d} - \theta = \frac{\lambda}{d^2},$$

which implies that θ is rational, and so the lemma is proved.

To prove Segre's theorem, we first locate θ between consecutive fractions a/b and c/d of F_1. If (2) holds, we apply the lemma. If (2) fails to hold, that is, if

$$\frac{a}{b} < \theta < \frac{a+c}{b+d} < \frac{c}{d},$$

choose the positive integer j so that

$$\frac{a}{b} < \frac{(j+1)a + c}{(j+1)b + d} < \theta < \frac{ja + c}{jb + d}.$$

Then apply the lemma with c/d replaced by (ja + c)/(jb + d). Thus we obtain one solution for (1).

Next choose n sufficiently large so that the two fractions of F_n adjacent to θ have denominators larger than the denominator of the solution of (1) already obtained. Thus we get, say, a_1/b_1 and c_1/d_1 in F_n with

$$\frac{a_1}{b_1} < \theta < \frac{c_1}{d_1}.$$

Repeating the process of the preceding paragraph, we get another solution of (1).

Then choose n sufficiently large so that the fractions in F_n adjacent to θ have larger denominators than the previous solutions to (1). This process can be continued indefinitely to produce infinitely many solutions to (1).

We conclude with an observation about the proof. It might appear that by using weighting factors, in the addition of the inequalities (4), we could alter (5) to obtain a result stronger than that of Segre, or at least different from it. However, it is not difficult to establish that nothing new can be obtained by such a procedure.

REFERENCES

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University of Oregon