INVARIANT LEVI FACTORS

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1. INTRODUCTION

Let L be a finite-dimensional Lie algebra over a field of characteristic zero. By a well-known theorem of Levi (see [8]), L is a vector-space direct sum of a semi-simple subalgebra S and the radical R, the latter being the maximal solvable ideal of L. S is called a Levi factor of L. In [7], we proved that if G is a finite group of automorphisms of L, then L contains Levi factors which are left fixed by the elements of G, in other words, G-invariant Levi factors. (In [7], the result is obtained for a wider class of algebras, and G is allowed to possess anti-automorphisms as well as automorphisms. However, since τ is an anti-automorphism of the Lie algebra L if and only if $-\tau$ is an automorphism, it suffices to assume that G consists of automorphisms of L).

We discuss here the uniqueness question for G-invariant Levi factors of L. Let N denote the nil radical of L, that is, the maximal nilpotent ideal in L. Then, if $z \in N$, the mapping Ad z of L defined by $b(\text{Ad } z) = [b, z]$ is a nilpotent derivation of L, and its exponential $\exp(\text{Ad } z)$ is an automorphism of L. It follows from the Campbell-Baker-Hausdorff formula ([1] or [2]), that the collection of these automorphisms for $z \in N$ forms a group. In [4] and [6], it is shown that if S and T are two Levi factors of L, then there exists an automorphism $\exp(\text{Ad } z)$ of L ($z \in N$) which maps S isomorphically onto T.

In [7], we conjectured that if S and T are two G-invariant Levi factors of L, then there exists an automorphism of L which commutes with each element of G and carries S onto T. Here we shall prove this conjecture (see Corollary 1), and we now proceed to describe the type of automorphism which will accomplish the result.

2. DEFINITIONS

We call an element $z \in L$ *G-symmetric* if $\tau z = z$ for all $\tau \in G$. The G-symmetric elements of L form a subalgebra of L. The G-symmetric elements in N form a nilpotent subalgebra of L.

Now let z be a G-symmetric element of L. If $x \in L$ and $\tau \in G$, then

$$(\tau x)\text{Ad } z = [\tau x, z],$$

$$\tau(x \text{ Ad } z) = \tau([x, z]) = [\tau x, \tau z] = [\tau x, z] .$$

Therefore Ad z commutes with the element of G. Hence, if z is a G-symmetric element of N, then $\exp(\text{Ad } z)$, which is a polynomial in Ad z, commutes with the elements of G. It follows that if S is any G-invariant subalgebra of L, then so is $S \exp(\text{Ad } z)$.

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Definition. If z is a G-symmetric element of N, we call the automorphism exp (Ad z) a G-*symmetry* of L. Two subalgebras of L will be called G-*symmetric* if there exists a G-symmetry of L which carries one onto the other.

Using the Campbell-Baker-Hausdorff formula, one can easily show that the automorphisms exp (Ad z) with z G-symmetric in N form a group. Hence, the relation of G-symmetry is an equivalence relation among the subalgebras of L and is also an equivalence relation among the G-invariant subalgebras of L.

Now let S be a G-invariant Levi factor of L; then L = S + R. If we apply exp (Ad z) to L, where z is a G-symmetric element of N, then, since R is invariant under all automorphisms of L, we obtain L = S exp (Ad z) + R, where S exp (Ad z) is G-invariant, semi-simple, and isomorphic to L/R, and has zero intersection with R. Hence S exp (Ad z) is a G-invariant Levi factor of L. The converse will follow from our theorem.

3. THE THEOREM

THEOREM. *Let* L *be a finite-dimensional Lie algebra over a field of characteristic zero. Let* R *denote the radical of* L, *and let* N *denote its nil radical. Let* G *be a finite group of automorphisms of* L. *Let* S *be a* G-*invariant semi-simple subalgebra of* L, *and let* L = T + R *be a Levi decomposition of* L, *where* T *is a* G-*invariant Levi factor of* L. *Then there exists a* G-*symmetric element* $z \in N$ *such that* $S \exp(\mathrm{Ad}\, z) \subset T$.

Proof. Let L' denote the derived algebra [L, L] of L, and let

$$L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$$

denote the terms of the derived series of L. Then, for some integer n, $R^{(n)} = \{0\}$, and also, since $N \subset R$, $N^{(n)} = \{0\}$.

If $s \in S$, then $s = t(s) + r(s)$, where $t(s) \in T$ and $r(s) \in R$. Let $s_1, s_2 \in S$. Then

$$[s_1, s_2] = [t(s_1), t(s_2)] + [t(s_1), r(s_2)] + [r(s_1), t(s_2)] + [r(s_1), r(s_2)].$$

Hence we have

$$t([s_1, s_2]) = [t(s_1), t(s_2)], \tag{1}$$

$$r([s_1, s_2]) = [t(s_1), r(s_2)] + [r(s_1), t(s_2)] + [r(s_1), r(s_2)]. \tag{2}$$

Also, since S, T and R are G-invariant, we have

$$t(\tau s) = \tau t(s), \quad r(\tau s) = \tau r(s) \qquad (s \in S,\ \tau \in G). \tag{3}$$

Since $[L, R] \subset N$, it follows from (2) that $r([s_1, s_2]) \in N$ for $s_1, s_2 \in S$. But since S is semi-simple, S = S'; and it follows that $r(s) \in N$ for all $s \in S$.

Hence, $S \subset T + N$. Let $G_0 = \exp(\mathrm{Ad}\, 0)$; that is, let G_0 be the identity mapping of L. Then $S^{G_0} \subset T + N^{(0)}$ (where $N^{(0)} = N$). Now suppose that we have found G-symmetries $G_0, G_1, \cdots, G_k$ such that

$$S_k = S^{G_0 G_1 \cdots G_k} \subset T + N^{(k)}.$$

We shall show that then there exists a G-symmetry G_{k+1} such that

$$S^{G_0 G_1 \cdots G_{k+1}} \subset T + N^{(k+1)}.$$

Since $N^{(n)} = \{0\}$, $G_0 G_1 \cdots G_n$ will be the desired G-symmetry.

If $s \in S_k$, then $S = t(s) + r(s)$, where $t(s) \in T$ and $r(s) \in N$. By the induction hypothesis, we take $r(s) \in N^{(k)}$. Then (1), (2), and (3) hold for s_1, s_2, $s \in S_k$ and $\tau \in G$.

Now consider $N^{(k)}/N^{(k+1)}$. We regard this as a representation module for S_k by defining $s \cdot \bar{z} = \overline{[t(s), z]}$ for $s \in S_k$, $z \in N^{(k)}$, $\bar{z} = z + N^{(k+1)} \in N^{(k)}/N^{(k+1)}$. By (2), we have

$$\overline{r([s_1, s_2])} = s_1 \cdot \overline{r(s_2)} - s_2 \cdot \overline{r(s_1)}.$$

Therefore, the mapping $s \to \overline{r(s)}$ of S into $N^{(k)}/N^{(k+1)}$ satisfies the conditions of a lemma of Whitehead's (see [5], Theorem 2.1). Since $H^1(S, N^{(k)}/N^{(k+1)}) = \{0\}$, (see [3], Theorem 25.1), there exists a $z \in N^{(k)}$ such that

$$\overline{r(s)} = s \cdot \bar{z} = \overline{[t(s), z]} \qquad (s \in S_k). \tag{4}$$

Now set

$$z' = \frac{1}{g} \sum_{\tau \in G} \tau z,$$

where g is the order of G. If $\rho \in G$, then

$$\rho z' = \frac{1}{g} \sum_{\tau \in G} (\rho\tau) z = z'.$$

Since $N^{(k)}$ is G-invariant, it follows that z' is a G-symmetric element of $N^{(k)}$. The elements of G induce mappings in $L/N^{(k+1)}$ by $\tau(\bar{l}) = \overline{\tau l}$ for $\tau \in G$ and $l \in L$. We now compute $s \cdot \overline{z'}$ for $s \in S_k$, z' as above. By direct computation one finds that

$$s \cdot \overline{z'} = \frac{1}{g} \sum_{\tau \in G} \overline{[t(s), \tau(z)]},$$

where by (3)

$$\overline{[t(s), \tau(z)]} = \overline{\tau[\tau^{-1} t(s), z]} = \overline{\tau[t(\tau^{-1} s), z]}.$$

Now, using (4) and (3), we obtain

$$\overline{[t(s), \tau(z)]} = \tau\overline{[t(\tau^{-1} s, z)]} = \overline{\tau r(\tau^{-1} s)} = \overline{\tau r(\tau^{-1} s)} = \overline{r(s)}.$$

Hence,

$$\overline{s \cdot z'} = \frac{1}{g} \sum_{\tau \in G} \overline{r(s)} = \overline{r(s)}.$$

and we have

$$r(s) = s \cdot \overline{z'} = \overline{[t(s), z']}. \tag{5}$$

Now set $z_k = -z'$. Then z_k is a G-symmetric element in $N^{(k)}$. Let G_{k+1} denote the G-symmetry $\exp(\mathrm{Ad}\, z_k)$. Then, if $s \in S_k$, we have

$$s^{G_{k+1}} = s[1 + \mathrm{Ad}(-z') + \cdots],$$

where the omitted terms involve two or more multiplications by $-z' \in N^{(k)}$ and hence yield elements in $N^{(k+1)}$. Hence,

$$\begin{aligned} s^{G_{k+1}} &\equiv s + [s, -z'] \pmod{N^{(k+1)}} \\ &= t(s) + r(s) + [t(s), -z'] + [r(s), -z'] \\ &\equiv t(s) + r(s) - [t(s), z'] \pmod{N^{(k+1)}} \\ &\equiv t(s) \pmod{N^{(k+1)}} \qquad \text{by (5)}. \end{aligned}$$

Thus, $S_k^{G_{k+1}} \subset T + N^{(k+1)}$, and this completes the proof of the theorem.

COROLLARY 1. *If* S *and* T *are any two* G-*invariant* Levi *factors of* L, *then* S *and* T *are* G-*symmetric*.

COROLLARY 2. *Any* G-*invariant semi-simple subalgebra of* L *may be embedded in a* G-*invariant Levi factor of* L.

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