SOLUTION OF A GEOMETRIC PROBLEM BY FEJES TÒTH

Heinrich Larcher

By "distance between two points on the unit sphere" or, in short, "distance between two points," we mean the length of the smaller of the two arcs of the great circle passing through the two points, and for two points that are diametrically opposite, one half of the circumference of a great circle. In [1], Fejes Tòth conjectured that the sum of the $\binom{s}{2}$ distances between s points on the unit sphere is at most $s^2\pi/4$ if s is even and at most $(s^2 - 1)\pi/4$ if s is odd. In the same paper, the author proved the conjecture for $s \leq 6$. The conjecture for even s was proved by Sperling in [2]. In this paper we shall show that the conjecture is true for odd s.

The first part of this paper is parallel to what Sperling did in [2], and we shall closely adhere to his notation. In the remainder of this paper, s is a positive odd integer. If $P_1, P_2, \cdots, P_s$ are s arbitrary points on the unit sphere (not necessarily distinct), we denote the distance between P_i and P_j by $\widehat{P_i P_j}$, and we put

$$E = \frac{1}{2} \sum_{i,j=1}^{s} \widehat{P_i P_j}.$$

If P_i' is the point diametrically opposite P_i, then one half of the sum of the $\widehat{P_i' P_j'}$ is E. Observing that the sum of the distances of any point from any two diametrically opposite points is π, we readily see that for the sum of the $\binom{2s}{2}$ distances between the 2s points $P_1, \cdots, P_s, P_1', \cdots, P_s'$ we have the equality

$$2E + \sum_{i,j=1}^{s} \widehat{P_i P_j'} = s^2\pi. \tag{1}$$

We denote the unit vectors from the center of the sphere to the points P_i and P_i' by x_i and x_i', respectively. For the inner product of the two vectors x_i and x_j we write (x_i, x_j). Henceforward, by "vector" we mean "unit vector."

If the angle α has the meaning indicated in Diagram 1, in which the diameters AA' and $P_i P_i'$ are perpendicular, then $\alpha = \arcsin(x_i, x_j)$. From Diagram 1 we see that

$$\widehat{P_i P_j'} = \widehat{P_i P_j} + 2\alpha = \widehat{P_i P_j} + 2 \arcsin(x_i, x_j),$$

which in turn implies that

$$\sum_{i,j=1}^{s} \widehat{P_i P_j'} = \sum_{i,j=1}^{s} \widehat{P_i P_j} + 2 \sum_{i,j=1}^{s} \arcsin(x_i, x_j) = 2E + 2 \sum_{i,j=1}^{s} \arcsin(x_i, x_j). \tag{2}$$

Received July 11, 1961.

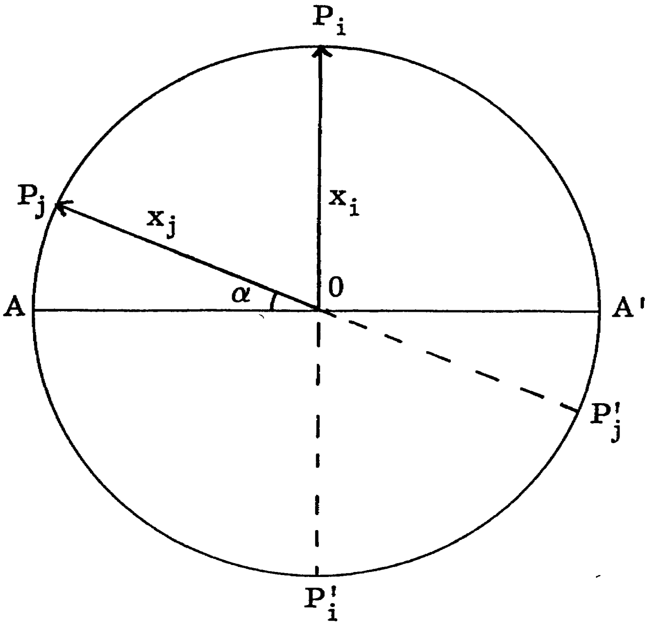

Diagram 1

Let $S_s = \{x_i | i = 1, \cdots, s\}$. Putting

$$\sum_{i,j=1}^{s} \arcsin(x_i, x_j) = h(x_1, \cdots, x_s)$$

$$= h(S_s)$$

and substituting (2) in (1), we obtain the equality

$$(3) \qquad 4E + 2h(S_s) = s^2 \pi .$$

The proof of the conjecture will be complete if we can show that for s arbitrary points on the unit sphere, $h(S_s) \geq \pi/2$. First we prove three lemmas exhibiting properties of the function $h(S_s)$.

LEMMA 1. *If* x *is any unit vector and* $x' = -x$, *then*

$$h(x_1, \cdots, x_s, x, x') = h(x_1, \cdots, x_s) .$$

Proof. Since the arcsine is an odd function, the contribution of x and x' to the sum h is

$$2 \sum_{i=1}^{s} \arcsin(x, x_i) + 2 \sum_{i=1}^{s} \arcsin(x', x_i) + 2 \arcsin(x, x') + \arcsin(x, x) + \arcsin(x', x')$$

$$= 2 \sum_{i=1}^{s} \arcsin(x, x_i) - 2 \sum_{i=1}^{s} \arcsin(x, x_i) - \pi + \frac{\pi}{2} + \frac{\pi}{2},$$

which is equal to zero. The next lemma follows immediately from the definition of $h(x_1, \cdots, x_s)$.

LEMMA 2. *For any* t *with* $1 \leq t < s$,

$$h(x_1, \cdots, x_s) = h(x_1, \cdots, x_t) + h(x_{t+1}, \cdots, x_s) + 2 \sum_{i=1}^{t} \sum_{j=t+1}^{s} \arcsin(x_i, x_j) .$$

If $h(x_1, \cdots, x_s)$ corresponds to a distribution of s points on the unit sphere, then for any positive integer n we write $h(nx_1, \cdots, nx_s)$ for the function h that corresponds to the distribution of ns points in which every point in the original distribution appears n times.

LEMMA 3. $h(nx_1, \cdots, nx_s) = n^2 h(x_1, \cdots, x_s)$.

Proof. The proof is by induction. The lemma holds for $n = 1$. Let us assume that it holds for $n - 1$: $h((n-1)x_1, \cdots, (n-1)x_s) = (n-1)^2 h(x_1, \cdots, x_s)$. Then by Lemma 2,

$$h(nx_1, \cdots, nx_s) = h((n-1)x_1, \cdots, (n-1)x_s, x_1, \cdots, x_s) = h((n-1)x_1, \cdots, (n-1)x_s)$$

$$+ h(x_1, \cdots, x_s) + 2(n-1)\sum_{i=1}^{s}\sum_{j=1}^{s} \arcsin(x_i, x_j)$$

$$= (n-1)^2 h(x_1, \cdots, x_s) + h(x_1, \cdots, x_s) + 2(n-1)h(x_1, \cdots, x_s)$$

$$= n^2 h(x_1, \cdots, x_s),$$

which was to be proved.

The minimum value of h for all possible distributions of s points we denote by $\min h(S_s)$. Clearly, $h(S_s)$ assumes an absolute minimum, since it is a continuous function in all its variables, and since its domain of definition is a compact space. Trivially, $\min h(S_1) = h(S_1) = h(x_1) = \pi/2$. Because of Lemma 1, we can write

$$(4) \qquad \frac{\pi}{2} = h(S_1) = \min h(S_3) = \min h(S_5) \geq \min h(S_7) \geq \cdots \geq \min h(S_s) \geq \cdots .$$

The first three equality signs in (4) are a consequence of (3) and of the proof of the conjecture by Fejes Tòth for $s \leq 6$. Our object is to show that equality holds throughout.

Let us assume that there is at least one strict inequality in (4); and, in order to be definite, let us consider the smallest odd integer s for which

$$\min h(S_s) = m < \frac{\pi}{2}.$$

Actually, m is positive. This fact can be derived rather easily from Sperling's work. We omit the derivation, since our proof does not make use of this fact.

We call two vectors x_i and x_j distinct if $x_i \neq x_j$. A set of distinct vectors is one in which no two are equal. A maximal subset of distinct vectors of a set of vectors is a subset of distinct vectors that cannot be enlarged to form a new subset of distinct vectors.

THEOREM 1. *Let* $\bar{S}_s = \{\bar{x}_i \mid i = 1, \cdots, s\}$ *be such that* $h(\bar{S}_s) = m$. *Then* $\bar{x}_1, \cdots, \bar{x}_t$ $(1 < t \leq s)$ *is a maximal subset of distinct vectors of* $\bar{S}_s$; *and if* $n_i \geq 1$ *is the number of vectors that coincide with* $\bar{x}_i$ $(i = 1, \cdots, t)$, *then*

$$(5) \qquad \sum_{\substack{i=1 \\ i\neq k}}^{t} n_i \frac{\bar{x}_k \times \bar{x}_i}{|\bar{x}_k \times \bar{x}_i|} = 0 \qquad (k = 1, \cdots, t),$$

where $\bar{x}_k \times \bar{x}_i$ *denotes the cross product of the vectors* $\bar{x}_k$ *and* $\bar{x}_i$ *while* $|\bar{x}_k \times \bar{x}_i|$ *denotes its magnitude.*

Proof. It follows from Lemma 1 and the fact that $h(\bar{S}_s) < \min h(S_{s-2})$ that $x_j \neq -x_i$ for $i, j = 1, \cdots, s$. Furthermore, not all s vectors in $\bar{S}_s$ can coincide. For if $x_i = x_1$ $(i = 2, \cdots, s)$, the sum E defined earlier is zero; and hence by (3), $h(\bar{S}_s) = s^2\pi > \pi/2$. Consequently, $t \geq 2$. Evidently, $\Sigma_{i=1}^{t} n_i = s$.

We put $h(\bar{S}_s) = \bar{h}(\bar{x}_1, \cdots, \bar{x}_t) = \Sigma_{i,j=1}^{t} n_i n_j \arcsin(\bar{x}_i, \bar{x}_j)$, where $x_j \neq \pm x_i$ for $j \neq i$ $(i, j = 1, \cdots, t)$. If we move any one of the t vectors, keeping its initial point fixed, $\bar{h}$ stays the same or increases. Let $x_i = (a_i, b_i, c_i)$, and let $U = \{x \mid (x, x) = 1\}$. Then $\bar{x}_1, \cdots, \bar{x}_t$ is a set of solution vectors for the problem of minimizing $\bar{h}(x_1, \cdots, x_t)$ subject to the condition that x_i be in U $(i = 1, \cdots, t)$. We solve the problem by Lagrange's multiplier method. Using multipliers λ_i, we put the gradient vectors $(\partial/\partial a_k, \partial/\partial b_k, \partial/\partial c_k)$ $(k = 1, \cdots, t)$ of

$$s\pi/2 + \sum_{\substack{i,j=1 \\ j\neq i}}^{t} n_i n_j \arcsin(a_i a_j + b_i b_j + c_i c_j) + \sum_{i=1}^{t} \lambda_i (a_i^2 + b_i^2 + c_i^2 - 1)$$

equal to zero. We eliminate the λ_i's, and we obtain the equations (5) which are necessary conditions $\bar{x}_1, \cdots, \bar{x}_t$ must satisfy.

For $k \neq i$ we define $a_{ki} = 1/|x_k \times x_i|$ (note that $x_k \neq \pm x_i$), and we define $a_{kk} = 1$ $(i, k = 1, \cdots, t)$.

THEOREM 2. *If the vectors* $x_1, \cdots, x_t$ $(t \geq 2;\ x_k \neq \pm x_i$ *for* $k \neq i)$ *satisfy the system of equations* (5), *then they lie in a plane.*

Proof. If $t = 2$, the theorem is trivial. Let $t > 2$, and let p_{12} denote the plane determined by x_1 and x_2. We form the inner product, indicated by a dot, of the k-th equation in (5) with x_1 and x_2, respectively. After applying the distributive law, we obtain the equations

$$\sum_{i=1}^{t} n_i a_{ki} (x_k \times x_i) \cdot x_1 = 0 \quad \text{and} \quad \sum_{i=1}^{t} n_i a_{ki} (x_k \times x_i) \cdot x_2 = 0 \quad (3 \leq k \leq t).$$

By multiplying the two equations by suitable constants and adding, we eliminate the terms $(x_k \times x_2) \cdot x_1$ and $(x_k \times x_1) \cdot x_2$, respectively. The resulting single equation can be written as

$$\left[\sum_{i=3}^{t} n_i a_{ki} (x_k \times x_i) \right] \cdot (n_1 a_{k1} x_1 + n_2 a_{k2} x_2) = 0. \tag{6}$$

Let X_k denote the first vector in the last inner product. Since the second vector is not the zero vector, equation (6) implies that either

i) $X_k = 0$ or

ii) $X_k \neq 0$, but X_k is perpendicular to $n_1 a_{k1} x_1 + n_2 a_{k2} x_2$, and hence is perpendicular to p_{12}. In case ii), x_k lies in p_{12}, since x_k is perpendicular to X_k. If i) holds, then by (5)

$$\sum_{i=3}^{t} n_i a_{ki} (x_k \times x_i) = 0 = -x_k \times (n_1 a_{k1} x_1 + n_2 a_{k2} x_2).$$

Since the last vector in the parentheses is not the zero vector,

$$n_1 a_{k1} x_1 + n_2 a_{k2} x_2 = c_k x_k$$

for a suitable constant $c_k \neq 0$; that is, x_k lies in the plane p_{12} $(3 \leq k \leq t)$. Therefore all vectors x_i lie in p_{12}.

However, for distributions all of whose points lie on a great circle we have the following theorem.

THEOREM 3. *If* P_i $(i = 1, \cdots, s)$ *are* s *arbitrary points on the unit circle, not necessarily distinct in position, then* $h(S_s) \geq \pi/2$.

Proof. This is more easily shown by working with E than with h; either will do, since the two are related by equation (3). We pick P_1 and rename the remaining points, if necessary, in such a way that $P_2, \cdots, P_s$ appear counterclockwise in this order. If two or more points coincide in position, we pick one of them and assign successive subscripts to the others. Let $P_{i+s} = P_{s+i} = P_i$. Evidently,

$$\sum_{i=1}^{s} \widehat{P_i P_{i+1}} \leq 2\pi .$$

In general,

$$\sum_{i=1}^{s} \widehat{P_i P_{i+k}} \leq 2\pi k \qquad (1 \leq k \leq (s-1)/2) .$$

To prove this, we observe that

$$\widehat{P_i P_{i+k}} \leq \widehat{P_i P_{i+1}} + \widehat{P_{i+1} P_{i+2}} + \cdots + \widehat{P_{i+k-1} P_{i+k}} \qquad (i = 1, \cdots, s), \tag{7}$$

since $\widehat{P_i P_{i+k}}$ is the smaller of the two arcs into which the unit circle is divided by P_i and P_{i+k}. It is understood that when P_i and P_{i+k} coincide, the smaller of the arcs is of zero length. For fixed i_0, $\widehat{P_{i_0} P_{i_0+1}}$ appears on the right-hand side of exactly k inequalities of (7); namely, of those for which $\widehat{P_{i_0-r+s} P_{i_0-r+k}}$ $(r = 0, 1, \cdots, k-1)$ is the left-hand side. Thus we have the inequality

$$\sum_{i=1}^{s} \widehat{P_i P_{i+k}} \leq k \sum_{i=1}^{s} \widehat{P_i P_{i+1}} \leq 2\pi k ,$$

as asserted above. From the statements

$$\widehat{P_i P_{i+k}} = \widehat{P_{i+k} P_i} = \widehat{P_{i+k} P_{i+s}} = \widehat{P_{i+k} P_{i+k+(s-k)}}$$

we infer that

$$\sum_{i=1}^{s} \widehat{P_i P_{i+k}} = \sum_{i=1}^{s} \widehat{P_{i+k} P_{i+k+(s-k)}} = \sum_{j=1}^{s} \widehat{P_j P_{j+(s-k)}} \qquad (1 \leq k \leq (s-1)/2) .$$

Hence,

$$\sum_{k=1}^{(s-1)/2} \sum_{i=1}^{s} \widehat{P_i P_{i+k}} = \sum_{k=(s+1)/2}^{s} \sum_{i=1}^{s} \widehat{P_i P_{i+k}}$$

and

$$E = \frac{1}{2}\sum_{j=1}^{s}\sum_{i=1}^{s} \widehat{P_i P_j} = \frac{1}{2}\left(\sum_{k=1}^{(s-1)/2}\sum_{i=1}^{s} \widehat{P_i P_{i+k}} + \sum_{k=(s+1)/2}^{s}\sum_{i=1}^{s} \widehat{P_i P_{i+k}}\right)$$

$$= \sum_{k=1}^{(s-1)/2}\sum_{i=1}^{s} \widehat{P_i P_{i+k}} \leq 2\pi(1 + 2 + \cdots + (s-1)/2) = (s^2 - 1)\pi/4.$$

Introducing the last expression in (3), we obtain the desired conclusion.

We have shown that if there is a strict inequality in (4), then that set of s points for which $\min h(S_s) = m < \pi/2$ must lie on a great circle. But, for such a set of points $h(S_s) \geq \pi/2$ by Theorem 3. This is a contradiction, and therefore the members of the sequence in (4) are all equal. We have thus proved the following proposition.

THEOREM 4. *If* P_i ($i = 1, \cdots, s$ *and* s odd) *are* s *arbitrary points on the unit sphere, then*

$$E = \frac{1}{2}\sum_{i,j=1}^{s} \widehat{P_i P_j} \leq (s^2 - 1)\pi/4.$$

Finally we shall determine which distributions satisfy the system of equations (5).

LEMMA 4. *For distributions that satisfy the hypothesis of Theorem* 2, n_1 *is an odd integer and* $n_i = n_1$ ($i = 2, \cdots, t$).

Proof. Let such a distribution be given. All its points lie on a great circle, which is drawn in Diagram 2. The vectors $(x_1 \times x_i)/|x_1 \times x_i|$ ($i = 2, \cdots, t$) are unit vectors perpendicular to the plane through the great circle. Since $\Sigma_{i=1}^{t} n_i = s$, the first equation of (5) will be satisfied if and only if $s - n_1$ is even and half of the vectors (each counted as often as n_i indicates) lie on one side of the diameter AA' in Diagram 2, and half of them lie on the other side. Thus n_1 is necessarily odd. Evidently, the same holds for $k = 2, \cdots, t$.

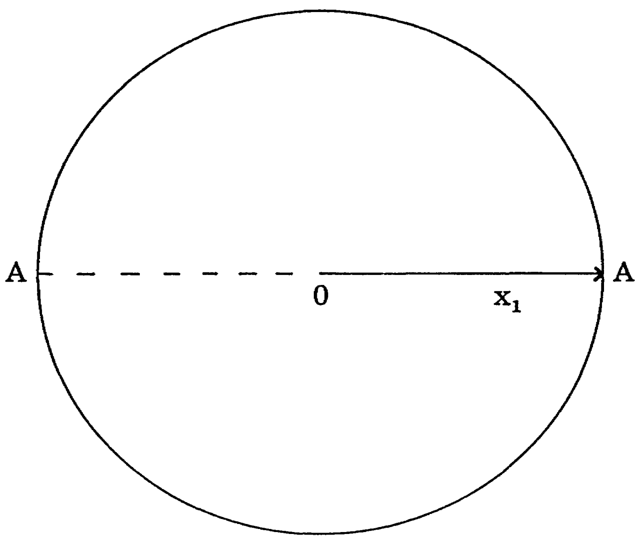

Diagram 2

In Diagram 2, we imagine the diameter AA' rotated about O counterclockwise until it coincides for the first time with a vector of our set, say x_{k_0}. Then x_{k_0} can only coincide with OA. For, it is easily seen that, if it were to coincide with OA', the k_0-th equation of (5) would not be satisfied. However, if x_{k_0} coincides with OA, the k_0-th equation is satisfied provided $n_{k_0} = n_1$. Continuing in this way, one proves the lemma.

THEOREM 5. *For a distribution of* s *points* (s *odd*) *for which the* *vectors* $x_1, \cdots, x_s$ *satisfy* (5), $E = (s^2 - 1)\pi/4$ *if and only if* $x_k \neq \pm x_i$ $(k \neq i$ *and* $i, k = 1, \cdots, s)$.

Proof. In order for (5) to hold, it is sufficient that $x_k \neq - x_i$. Let $n = n_1$ in Lemma 4, and let us assume that $n > 1$. By Lemma 3, we have the relations

$$h(nx_1, \cdots, nx_t) = n^2 h(x_1, \cdots, x_t) \geq n^2 \pi/2 > \pi/2 .$$

On the other hand, if all vectors are distinct, that is, if $x_k \neq x_i$ for $k \neq i$, then $n = 1$; and (5) implies equality throughout (7). This in turn implies that $E = (s^2 - 1)\pi/4$.

All distributions of s points for which $E = (s^2 - 1)\pi/4$ or $h(S_s) = \pi/2$ are now known. They have the property that, for some t $(0 \leq 2t < s)$, $2t$ of the s points can be paired so that each pair consists of a point and its diametrically opposite point; while for the remaining $s - 2t$ points, the vectors $x_1, \cdots, x_{s-2t}$ are all distinct, lie in a plane, and have the additional property that a diameter coinciding with any one of these vectors divides them in such a way that $(s - 2t - 1)/2$ vectors lie on one side of the diameter and $(s - 2t - 1)/2)$ on the other.

REFERENCES

1. L. Fejes Tòth, *Über eine Punktverteilung auf der Kugel,* Acta Math. Acad. Sci. Hungar. 10 (1959), 13-19.

2. G. Sperling, *Lösung einer elementargeometrischen Frage von Fejes Tòth,* Arch. Math. 11 (1960), 69-71.

Michigan State University