BOUNDARY FUNCTIONS FOR FUNCTIONS DEFINED IN A DISK

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1. INTRODUCTION

Let C and D denote respectively the unit circle $|z| = 1$ and the open unit disk $|z| < 1$ in the complex plane. By an *arc at* $\zeta \in C$ we mean a Jordan arc that lies in D except for one end point at ζ. Let $\phi(\zeta)$ and f(z) be real- or complex-valued functions defined on C and D, respectively. (For real-valued functions, we admit $+\infty$ and $-\infty$ as values; for complex-valued functions, we admit ∞ as a value. Instead of the real or complex numbers, we could consider values in more general spaces, but it is doubtful that such extensions would enhance the intrinsic value of our theorems.) We shall say that ϕ is a *boundary function* for f, or that f has a boundary function ϕ, provided that for each $\zeta \in C$ there exists an arc $A(\zeta)$ at ζ such that

$$\lim_{z \to \zeta,\ z \in A(\zeta)} f(z) = \phi(\zeta).$$

When we speak simply of a function f, no restrictions whatever (such as analyticity or continuity, for example) are assumed, unless they are explicitly stated.

In Section 2, we consider the problem of how many different boundary functions ϕ a particular function f can have. Section 3 is concerned with the relation between boundary functions and the Baire classification. In Section 4, finally, we pose a number of problems.

2. THE NUMBER OF BOUNDARY FUNCTIONS POSSESSED BY A FUNCTION

If f is defined in D and if there exist two arcs A and A' at $\zeta \in C$ along which f(z) tends to two distinct limits b and b', respectively, as $z \to \zeta$, we say that ζ is an *ambiguous point* of f. We shall make repeated use of the following fundamental result (see [1, p. 382, Corollary 1]).

THEOREM A. *No function defined in* D *has uncountably many ambiguous points on* C.

We apply this to obtain a theorem on unrestricted functions, and then give examples of various functions having fairly many boundary functions.

THEOREM 1. *Every function* f *in* D *has at most* $2^{\aleph_0}$ *boundary functions.*

Proof. By Theorem A, f has at most $\aleph_0$ ambiguous points. At each ambiguous point, f has at most $2^{\aleph_0}$ asymptotic values. Hence, f has at most $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$ boundary functions.

Received April 15, 1961.

This paper was written with partial support from the Air Force Office of Scientific Research, under Contract No. AF 49(638)-633.

COROLLARY 1. *If* $2^{\aleph_0} = \aleph_\beta$, $\alpha \geq \beta$, *and* $\mathscr{F}$ *is a class of* $\aleph_\alpha$ *functions defined in* D, *then there exist at most* $\aleph_\alpha$ *functions on* C *that are boundary functions for functions in* $\mathscr{F}$.

Proof. Since each function f in $\mathscr{F}$ has at most $\aleph_\beta$ boundary functions, there exist at most $\aleph_\alpha \cdot \aleph_\beta = \aleph_\alpha$ boundary functions for the functions in $\mathscr{F}$.

The example $v = \Im \log(1 - z)$ shows that a bounded harmonic function can have $2^{\aleph_0}$ different boundary functions. On the other hand, if f is a normal meromorphic function [5, p. 53] and b is an asymptotic value for f at $\zeta \in C$, then [5, p. 53, Theorem 2] f has the angular limit b at the point ζ. Therefore each normal meromorphic function in D has at most one boundary function. The following theorem implies that this result can not be transferred to the class of holomorphic functions of bounded characteristic.

THEOREM 2. *There exists a holomorphic function that belongs to every Hardy class* H_p $(0 < p < \infty)$ *and has* $2^{\aleph_0}$ *boundary functions.*

Proof. Gehring [2, pp. 287-288] has constructed a holomorphic function f belonging to every Hardy class H_p and having the asymptotic values 0 and ∞ at $z = 1$. If $a_n \to 0$ rapidly enough, then the function

$$g(z) = \sum_1^\infty a_n f(ze^{i/n})$$

has all the required properties.

3. BOUNDARY FUNCTIONS AND THE BAIRE CLASSIFICATION

There exist $2^{\aleph_0}$ Baire functions defined in D [3, p. 319, Theorem I], and by Corollary 1 there exist only $2^{\aleph_0}$ boundary functions for this class. It would be interesting to know if all these boundary functions are Baire functions.

CONJECTURE. *If* f *is a function of Baire class* β *and has a boundary function* ϕ, *then* ϕ *is a function of Baire class at most* $\beta + 2$.

We do not know even whether the conjecture is true for the case $\beta = 0$; for a result related to the conjecture, see Theorem 8.

Every function ϕ defined on C is the boundary function for some function f defined in D (for example, let $f(0) = 0$ and $f(re^{i\theta}) = r\phi(e^{i\theta})$ for $r \neq 0$). Consequently not every boundary function is a Baire function. We can, however, make the following assertion.

THEOREM 3. *If the function* f *has a boundary function* ϕ *that is a Baire function, then every boundary function for* f *is a Baire function. If* ϕ *is of Baire class* $\alpha \geq 3$, *then every boundary function for* f *is of Baire class* α.

Proof. Let ϕ be of class α, and suppose that ϕ_1 is another boundary function for f. By Theorem A, ϕ_1 differs from ϕ at no more than countably many points, so that the function ϕ_1 is of Baire class β, where $\beta \leq \max(2, \alpha)$ (see [3, p. 352, Theorem VII]). By a similar argument, $\alpha \leq \max(2, \beta)$. This completes the proof.

THEOREM 4. *There exists a Baire function* ϕ *of class* 2 *such that, if* ϕ *is a boundary function for the function* f, *then every boundary function for* f *is of Baire class* 2.

Proof. There exists [3, pp. 368-369] a Baire function ϕ of class 2 that cannot be transformed into a Baire function of class 1 or 0 by altering its values at not more than countably many points. Suppose that ϕ is a boundary function for the function f. If ϕ_1 is a boundary function for f, then, by Theorem A, ϕ_1 can be obtained from ϕ by altering the values of ϕ at not more than countably many points. This implies that ϕ_1 is a Baire function of class at most 2, and, because of the nature of ϕ described above, ϕ_1 is of class at least 2.

THEOREM 5. *There exists a bounded continuous function having boundary functions of Baire classes* 0, 1, *and* 2, *respectively.*

Proof. Let $\zeta_n = e^{i\theta_n}$, where $\{\theta_1, \theta_2, \cdots\}$ is the set of rational numbers in the interval $[0, 2\pi]$; denote by $\{D_n\}$ a set of mutually disjoint open disks in D, with the property that D_n is tangent to C at ζ_n. Let C_n denote the boundary of D_n, and R_n the radius of D_n that terminates at ζ_n. Inside D_n, define f by the formula

$$f(z) = \frac{\operatorname{dist}(z, C_n)}{\operatorname{dist}(z, C_n) + \operatorname{dist}(z, R_n)}.$$

Outside the set $\bigcup D_n$, let $f(z) = 0$. Since every point of C is accessible along a path in D that meets none of the disks D_n, and since $f(z) \equiv 1$ on each of the segments R_n, the characteristic function of each subset of $\{\zeta_n\}$ is a boundary function for f, and the assertion of the theorem follows immediately (see [3, p. 365]).

THEOREM 6. *There exists a harmonic function having boundary functions of Baire classes* 0, 1, *and* 2, *respectively.*

Proof. Let μ denote the elliptic modular function defined and holomorphic in D, and let

$$f(z) = \Im \mu(z).$$

Then f is harmonic in D.

If $\zeta \in C$ is a cusp of the modular figure, and if A is an arc at ζ lying in the interior of a triangle of the modular figure, then, as $z \to \zeta$ along A, $\mu(z)$ tends to 0, 1, or ∞; we call ζ a cusp of the first, second, or third kind, according as the limit is 0, 1, or ∞. If ζ is of the first or second kind, then $f(z) \to 0$ as $z \to \zeta$ along A. If ζ is of the third kind, we can choose an arc A at ζ such that f(z) is identically equal to one of the two values ± 1 along A; also, we can choose an arc A' at ζ such that $f(z) \equiv 0$ on A'. We note that the countable set of cusps of the third kind is everywhere dense on C.

If $\zeta \in C$ is not a cusp of the modular figure, then there exists a sequence $\{T_1, T_2, \cdots\}$ of distinct triangles of the figure, with the properties that T_n has a side in common with T_{n+1} $(n = 1, 2, \cdots)$ and that $T_n \to \zeta$ as $n \to \infty$. Along the open sides of T_n, $\mu(z)$ is real, and hence $f(z) \equiv 0$. If z' and z" are points on distinct open sides of T_n, then, for every positive ε, it is possible to join z' to z" by means of an arc A_n which, except for its end points, lies in T_n, and on which $|f(z)| < \varepsilon$. Hence 0 is an asymptotic value for f at ζ, and an argument analogous to that concluding the proof of Theorem 5 is again applicable.

If f is defined in D and has a boundary function ϕ, and if $\mathscr{A}$ is a family of arcs (lying in D except for one end point) such that each point ζ on C is the end point of precisely one of the arcs $A(\zeta)$ in $\mathscr{A}$, and such that $f(z) \to \phi(\zeta)$ as $z \to \zeta$ along $A(\zeta)$, we say that f and ϕ *admit the family* $\mathscr{A}$ *of arcs.*

We call a function of Baire class 2 an *honorary function of Baire class* 2 if there exists a function of Baire class 1 differing from it at the points of only a countable set.

LEMMA 1. *Let* $\phi_n \to \phi$, *where each* ϕ_n *has at most countably many discontinuities. Then* ϕ *is at most an honorary function of Baire class* 2.

Proof. Denote by E_n the set of discontinuities of ϕ_n. Then the set $E = \bigcup E_n$ is at most countable and is therefore an F_σ, so that $M = C \setminus E$ is a set of type G_δ. Each ϕ_n is continuous on M, and hence, if ϕ^* denotes the restriction of ϕ to M, ϕ^* is of Baire class at most 1 on M. According to [4, p. 309], ϕ^* can be extended to a function ψ that is of Baire class at most 1 on C. Since ψ differs from ϕ only at the points of an at most countable set E, we have the conclusion of our lemma.

LEMMA 2. *Let* ϕ *be at most an honorary function of Baire class* 2. *Then* $\phi = \lim \phi_n$, *where each* ϕ_n *has at most a finite number of discontinuities, every one of which is a jump.*

Proof. The conclusion is obvious in case ϕ is of Baire class at most 1, so that we may assume ϕ to be of Baire class 2. By hypothesis, $\phi = \psi + \chi$, where $\chi = 0$ except at the points of an at most countable set $E = \{\zeta_1, \zeta_2, \cdots\}$, and $\psi = \lim \psi_n$, where each ψ_n is a continuous function. Define χ_n to be zero except at the points $\zeta_1, \cdots, \zeta_n$, where χ_n is made to coincide with χ, and write $\phi_n = \psi_n + \chi_n$. Then ϕ_n is continuous on C except at the points $\zeta_1, \cdots, \zeta_n$, where ϕ_n has jump discontinuities, and evidently $\phi_n \to \phi$, so that the lemma is proved. (In case some functional values are infinite, a slight, obvious modification of the foregoing argument is required.)

THEOREM 7. *If* f *is continuous in* D *and has a boundary function* ϕ, *and if* f *and* ϕ *admit a family of mutually disjoint arcs, then* ϕ *is at most an honorary function of Baire class* 2.

Proof. For $n = 1, 2, \cdots$, let C_n denote the circle $|z| = n/(n+1)$. Without loss of generality, we may assume that f and ϕ admit a family of mutually disjoint arcs $\{A(\zeta)\}$ each member of which meets each of the circles C_n. For each ζ and each n, let $\zeta^{(n)}$ denote the last point of $A(\zeta)$ on C_n, as one proceeds along $A(\zeta)$ towards C; let S_n denote the set of all points $\zeta^{(n)}$. Clearly the ordering of the set $S_n = \{\zeta^{(n)}\}$ on the circle C_n is the same as the cyclic ordering of the set $\{\zeta\}$ on the unit circle C. For all ζ on C, we write $\phi_n(\zeta) = f(\zeta^{(n)})$.

The set of points of S_n that are not two-sided limit points of S_n is at most countable [3, p. 177, Theorem I], and therefore ϕ_n has at most countably many points of discontinuity. Since $\phi_n \to \phi$, the conclusion of our theorem follows from Lemma 1.

THEOREM 8. *For* ϕ *to be a boundary function for some continuous function, it is sufficient that* ϕ *be at most an honorary function of Baire class* 2.

Proof. According to Lemma 2, $\phi = \lim \phi_n$, where each ϕ_n has at most a finite number of discontinuities, every one of which is a jump.

Let $\{\zeta_{1k}\}$ denote the set of points on C at which ϕ_1 is discontinuous; for $n = 2, 3, \cdots$, define $\{\zeta_{nk}\}$ to be the union of the set $\{\zeta_{n,k-1}\}$ with the set of points on C at which ϕ_k is discontinuous. For each n, we suppose that the indices k are chosen in accordance with the anticlockwise cyclic ordering of the set $\{\zeta_{nk}\}$ on C; and if the set $\{\zeta_{nk}\}$ contains exactly h points, we shall write $\zeta_{n,h+1} = \zeta_{n1}$.

Let $\{D_{1k}\}$ denote a set of circular disks in D, tangent to C at the points of the set $\{\zeta_{1k}\}$. We suppose that the disks have a common radius ρ_1, and that ρ_1 is so small that no two of the disks have a common boundary point. When the set of disks

$\{D_{n-1,k}\}$ has been defined, we choose the disks D_{nk} with radius ρ_n ($\rho_n < \rho_{n-1}/2$) tangent to C at the points ζ_{nk}, except that if $\zeta_{nk} = \zeta_{n-1,h}$, then D_{nk} shall be identical with $D_{n-1,h}$. At each stage, the radius ρ_n is chosen so small that no two of the disks D_{nk} have a common boundary point.

Let C_n^* denote the circle $|z| = 1 - \rho_n/2$. Let C_{nk} denote the open arc from ζ_{nk} to $\zeta_{n,k+1}$ (in the anticlockwise direction), and let C_{nk}^* denote the corresponding arc of C_n^* which lies between $\bar{D}_{nk}$ and $\bar{D}_{n,k+1}$. For each n, let g_{nk} denote a direction-preserving homeomorphism from C_{nk}^* to C_{nk}.

We are now ready to construct a continuous function f whose boundary function is ϕ. On C_{nk}^*, we define our function by the rule

$$f(z) = \phi_n(g_{nk}(z)).$$

On the radius of D_{nk} which terminates at ζ_{nk}, we use the definition

$$f(z) = \phi(\zeta_{nk}).$$

There is no difficulty now in extending f continuously to the whole of D in such a manner that ϕ is a boundary function for f.

We consider next those functions defined on C that are both characteristic functions of a point set and boundary functions for a continuous function.

THEOREM 9. *In order that the characteristic function of a set* E *on* C *be the boundary function for some continuous function, it is necessary and sufficient that both* E *and* $C \setminus E$ *be the union of a countable set and a set of type* G_δ.

Necessity. Suppose that the characteristic function ϕ of E ($E \subset C$) is a boundary function for a continuous function f. Let M_0 and M_1 denote the sets in D where the real part of f lies in the intervals (-1/4, 1/4) and (3/4, 5/4), respectively. The boundary of each component M_{0i} of M_0 meets C in a closed set. The set of points on C that are not boundary points of some component M_{0i} is therefore a set of type G_δ.

It follows that the set of points ζ on C for which $\phi(\zeta) = 1$ is the union of a set of type G_δ and a set of points on C that are simultaneously end points of arcs in M_0 and end points of arcs in M_1; and by means of Theorem A, it can easily be shown that the latter set is at most countable. A similar argument shows that the set of points where $\phi(\zeta) = 0$ has the same structure, and the necessity of the condition is thus established.

Sufficiency. Suppose that C is the union of the disjoint sets E_0, E_1, and E_2, with E_0 and E_1 of type G_δ, and E_2 finite or countable.

We draw a set of open disks D_i internally tangent to the unit circle C at the points ζ_i of E_2, in such a way that $\bar{D}_i \cap \bar{D}_j$ is empty for $i \neq j$. At each point ζ_i of E_2, we draw two rectilinear segments L_{i0} and L_{i1} that lie in D_i except for their common end point ζ_i.

Let B denote the set $\bar{D} \setminus \bigcup D_i$, and let w(z) be a function defined on B in such a way that

(i) $w(e^{i\theta}) = e^{i\theta}$,

(ii) $|w(z)| = |z|$,

(iii) w is continuous on B and univalent in the interior of B,

(iv) the image of B under w is the closed disk $|w| \leq 1$, and the image of the boundary of D_i lies on the radius terminating in the point $w = \zeta_i$.

In other words, let the mapping w "close the holes" that the removal of the disks D_i leaves in D.

We shall first define a function g, continuous in the disk $|w| < 1$, which has the radial limits 0 and 1, respectively, at each point of $w(E_0)$ and $w(E_1)$.

Since the set E_0 is of type G_δ, its image under w has the form

$$w(E_0) = \bigcap G_j \qquad (G_j \supset G_{j+1},\ G_j \text{ open}).$$

For each j, we divide each component of G_j into the union of adjacent closed (circular) intervals I_{jp} in such a way that the set of limit points of the points of division consists of the two end points of the component. For each of the closed intervals thus constructed, we refer to the two intervals adjacent to it as the *neighboring intervals*. We then apply a similar construction to the set $w(E_1)$, obtaining closed intervals J_{jp}, and we order all the closed intervals I_{jp} and J_{jp} obtained in the two constructions into a simple sequence $\{I_k\}$ of intervals on C (if some interval occurs s times in $\{I_{jp}\}$ and t times in $\{J_{jp}\}$, it shall occur $s + t$ times in $\{I_k\}$).

For $k = 1, 2, \cdots$, we denote by C_k the circle $|w| = 1 - 2^{-k}$. On the arc I_k^* of C_k whose projection from the origin is the interval I_k on C, we define $g(w)$ to have the value 0 or 1, according as the interval I_k arose from the set $w(E_0)$ or the set $w(E_1)$.

On each radius of the disk $|w| < 1$, we define the function $g(w)$ in accordance with the following program.

1. On all radii that do not meet the interval I_1^* or its two neighboring intervals, $g(w) = 0$ between the origin and the circle C_1. On the two neighboring intervals of I_1^*, g is a linear function (of arg w), chosen so as to provide continuity; and on the radial segments corresponding to I_1^* and its two neighboring intervals, g is a continuous linear function of $|w|$.

2. Suppose that $g(w)$ has been defined for $|w| \leq 1 - 2^{1-k}$. On each radius that meets neither I_k^* nor its two neighboring intervals, g is constant between the circles $|w| = 1 - 2^{1-k}$ and $|w| = 1 - 2^{-k}$; on the neighboring intervals of I_k^*, g is a linear function (of arg w), chosen so as to provide continuity; and in the intersection of the annulus $1 - 2^{1-k} \leq |w| \leq 1 - 2^{-k}$ with the sector determined by I_k^* and its two neighboring intervals, g is a continuous linear function of $|w|$.

Clearly, each point of C lies either in at most finitely many of the intervals I_k (or neighbors of such intervals) that arise from E_0, or else in at most finitely many of the intervals I_k (or neighbors of such intervals) that arise from E_1. It follows that the function $g(w)$ has the radial limit 0 at each point of $w(E_0)$, and the radial limit 1 at each point of $w(E_1)$.

In $B \cap D$, we now define f by the formula $f(z) = g(w(z))$. The function f can easily be extended to the remainder of D in such a way that it takes the values 0 and 1 on L_{i0} and L_{i1}, respectively ($i = 1, 2, \cdots$), and so that it is continuous in D. This completes the proof.

THEOREM 10. *Not every function of Baire class 2 is a boundary function for some continuous function.*

Proof. Let E be the union of countably many perfect nowhere dense sets on C, and let E be everywhere dense on C; then the characteristic function ϕ of E is of Baire class 2 (see [3, pp. 368-369]). If ϕ were a boundary function for a continuous function, then, according to Theorem 8, E would have to be the union of a countable set and a set of type G_δ. Since E is locally uncountable, the set of type G_δ would have to be dense, and hence residual, on C. This, however, is impossible, since E is of first category. The proof is complete.

4. OPEN QUESTIONS

For the class of functions continuous in the disk D, the most important unsolved problem is the characterization of the functions ϕ on C that are boundary functions.

Problem 1. If ϕ is of Baire class 2 and is a boundary function for some function continuous in D, is ϕ merely an honorary function of Baire class 2?

Problem 2. In case the answer to Problem 1 is negative, does it become positive under the additional hypothesis that ϕ is the characteristic function of some point set on C?

Problem 3. If f is continuous in D and has a boundary function ϕ, do f and ϕ admit a family of mutually disjoint arcs?

Problem 4. In case the answer to Problem 3 is negative, is ϕ a boundary function for some continuous function g such that g and ϕ admit a family of mutually disjoint arcs?

Problem 5. Does there exist a bounded harmonic function having a boundary function of Baire class 2? We note that there does not exist a bounded holomorphic function, or even a normal meromorphic function, possessing a boundary function of Baire class 2; for if a normal meromorphic function f has a boundary function ϕ, then $\phi(\zeta)$ is the radial limit of f at ζ, and therefore ϕ is of Baire class 0 or 1.

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