SUMMABILITY AND ASSOCIATIVE INFINITE MATRICES

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We consider the sequence-to-sequence matrix transformations $y = Ax$, where $A = (a_{nk})$, $x = \{x_k\}$, $y = \{y_n\}$,

$$y_n = A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \qquad (n, k = 0, 1, 2, \cdots).$$

It is known that a matrix A is *conservative*, that is, Ax converges whenever x does, if and only if $\|A\| = \sup_n \Sigma_{k=0}^{\infty} |a_{nk}|$ is finite, $\lim_n \Sigma_{k=0}^{\infty} a_{nk}$ exists, and $\lim_n a_{nk}$ exists for $k = 0, 1, 2, \cdots$. If $A, B, C \cdots$ are conservative matrices with elements $a_{nk}, b_{nk}, c_{nk}, \cdots$, the column limits will be denoted by $a_k, b_k, c_k, \cdots$. A conservative matrix A is said to be *co-regular* if

$$\lim_n \sum_k a_{nk} - \sum_k a_k \neq 0.$$

Otherwise it is said to be *co-null*. Let e and e^n $(n = 0, 1, 2, \cdots)$ be the sequences defined respectively by $e_k = 1$ $(k = 0, 1, \cdots)$ and by $e_k^n = \delta_{nk}$ $(n, k = 0, 1, 2, \cdots)$. Let $\Delta = \{e^n : n = 0, 1, 2, \cdots\}$, and let Φ be the set consisting of the elements of Δ together with e. Let $H(\Delta)$ and $H(\Phi)$ be the linear hulls of Δ and Φ, respectively. The terms "basis" and "biorthogonal" will be used as in [1, pp. 106, 110].

We shall say that a matrix A is *associative* if $B(Ax) = BA(x)$ for all matrices B with $\|B\|$ finite and all x in the summability field C_A. Clearly a matrix A is associative if and only if

$$\sum_n t_n \sum_k a_{nk}x_k = \sum_k \sum_n t_n a_{nk}x_k \qquad \text{for all } x \in C_A \text{ and all } \{t_n\} \in (\gamma),$$

where (γ) denotes the set of sequences $\{t_n\}$ such that $\Sigma_{n=0}^{\infty} |t_n|$ converges. We shall show that if A is replaceable, that is, if there exists a regular matrix D such that $C_D = C_A$, then A is associative if and only if Φ is a basis for C_A.

Bases for the space C_A have been studied by Wilansky [3] and MacPhail [2]. A conservative matrix A is said to have *maximal inset* if $\Sigma a_k x_k$ converges for all x in C_A. A is said to have *propagation of maximal inset* (PMI) if $\Sigma b_k x_k$ converges for all x in C_A whenever B is a matrix such that $C_B = C_A$. Wilansky has shown that if A is a triangular co-regular matrix, then Φ is a basis for C_A if and only if A has PMI. MacPhail has shown that this statement is true if "triangular" is replaced by "reversible." We shall show that if A is an arbitrary co-regular matrix, then Φ is a basis for C_A if and only if A has PMI. Also, we shall give necessary and sufficient conditions that Δ be a basis for C_A.

LEMMA 1. *Let A be a co-regular matrix. $\overline{H(\Phi)} = C_A$, that is, $H(\Phi)$ is dense in C_A, if and only if, for each sequence $\{b_n\}$ such that $\Sigma |b_n|$ is convergent and*

Received September 6, 1960.

(1) $$\sum_k \sum_n b_n a_{nk} x_k \textit{ converges for all } x \textit{ in } C_A,$$

we have

(2) $$\sum_k \sum_n b_n a_{nk} x_k = \sum_k a_{nk} x_k \textit{ for all } x \textit{ in } C_A.$$

Proof. From [4] we know that $\overline{H(\Phi)} = C_A$ if and only if each continuous linear functional on C_A which is zero on Φ is zero on C_A. Denote the set of continuous linear functionals on C_A by C_A^*. From [4], if $f \in C_A^*$, then

(3) $$f(x) = b \lim_n A_n(x) + \sum_n b_n A_n(x) + \sum_n \alpha_k x_k \qquad (x \in C_A),$$

with $\{b_n\} \in (\gamma)$. Conversely, if $\Sigma \alpha_k x_k$ converges for all x in C_A, $\{b_n\} \in (\gamma)$, and b is arbitrary, then the function f defined by (3) is in C_A^*. Suppose $f \in C_A^*$ and f is zero on Φ. In a representation (3) for f, let x be successively e, e^0, e^1, $\cdots$; then

(4) $$b \lim_n \sum_k a_{nk} + \sum_n b_n \sum_k a_{nk} + \sum_k \alpha_k = 0,$$

(5) $$b a_k + \sum_n b_n a_{nk} + \alpha_k = 0 \qquad (k = 0, 1, 2, \cdots).$$

Summing (5) on k and subtracting from (4), we see that $b = 0$ since A is co-regular. The change of the order of summation in (4) is permissible, since $\Sigma_n b_n \Sigma_k a_{nk}$ is absolutely convergent. Thus $\alpha_k = -\Sigma_n b_n a_{nk}$ $(k = 0, 1, 2, \cdots)$, and

$$f(x) = \sum_n b_n \sum_k a_{nk} x_k - \sum_k \sum_n b_n a_{nk} x_k.$$

Now $\{b_n\} \in (\gamma)$ and satisfies (1), and therefore, if (2) holds, $f(x) = 0$ for all $x \in C_A$ and hence $\overline{H(\Phi)} = C_A$. Conversely, suppose that $H(\Phi) = C_A$, and that $\{b_n\} \in (\gamma)$ and satisfies (1). Letting

$$f(x) = \sum_n b_n \sum_k a_{nk} x_k - \sum_k \sum_n b_n a_{nk} x_k \qquad (x \in C_A),$$

we see that $f \in C_A^*$. But f is zero on Φ and hence on C_A, hence (2) holds.

Lemma 1 is clearly equivalent to the following.

LEMMA 1'. *Let* A *be co-regular.* $\overline{H(\Phi)} = C_A$ *if and only if* $B(Ax) = BA(x)$, *for all* x *in* C_A *and all* B *with* $\|B\|$ *finite and such that* $BA(x)$ *exists for all* x *in* C_A.

The following lemma is an immediate consequence of [3, Lemma 13].

LEMMA 2. *Let* A *be conservative,* $x \in C_A$. *If* $s(t_n) = \Sigma_k \Sigma_n t_n a_{nk} x_k$ *is convergent for all* $\{t_n\} \in (\gamma)$ *then* $s(t_n) = \Sigma_n t_n \Sigma_k a_{nk} x_k$.

Thus a conservative matrix A is associative if and only if $\Sigma_k \Sigma_n t_n a_{nk} x_k$ is convergent for all $\{t_n\} \in (\gamma)$ and $x \in C_A$.

An element x of an FK-space E (with $\Delta \subset E$) is said to have FAK *(funktionale Abschnittskonvergenz)* if $\Sigma_k x_k f(e^k)$ converges for each $f \in C_E^*$. If each element of E has FAK, E is said to have FAK.

LEMMA 3. *Let* A *be conservative. An element* y *in* C_A *has* FAK *if and only if*

$$\sum_k \sum_n t_n a_{nk} y_k \quad \text{converges for all } \{t_n\} \in (\gamma) \tag{6}$$

and

$$\sum_k a_k y_k \text{ converges.} \tag{7}$$

Proof. Let y have FAK, $\{t_n\} \in (\gamma)$. Let $f(x) = \Sigma_n t_n A_n(x)$ and $g(x) = \lim_n A_n(x)$. Since f and g are in C_A^*, (6) and (7) hold. Conversely, if (6) and (7) hold, let $f \in C_A^*$,

$$f(x) = b \lim_n A_n(x) + \sum_n t_n \sum_k a_{nk} x_k + \sum_k \alpha_k x_k \qquad (x \in C_A),$$

with $\{t_n\} \in (\gamma)$. Setting $x = e^k$ $(k = 0, 1, 2, \cdots)$, we see that $\Sigma_k y_k f(e^k)$ converges.

LEMMA 4. *Let* A *be a conservative matrix such that* Φ *is a basis for* C_A. *Then* C_A *has* FAK, *and consequently* A *is associative and has* PMI.

Proof. We have $x = \alpha e + \Sigma_k \alpha_k e^k$ $(x \in C_A)$, where α and α_k $(k = 0, 1, 2, \cdots)$, are uniquely defined for each $x \in C_A$. It may be shown by an argument similar to that in [1, p. 111] that the function h defined by $h(x) = \alpha$ $(x \in C_A)$ is linear and continuous. Also, $h(e) = 1$ and $h(e^k) = 0$ $(k = 0, 1, 2, \cdots)$. For each k, let

$$g_k(x) = x_k - h(x).$$

Then $g_k \in C_A^*$ for each k, and the system $\{h, g_0, \cdots; e, e^0, \cdots\}$ is biorthogonal. Therefore

$$x = h(x)e + \sum_k g_k(x) e^k = h(x)e + \sum_k (x_k - h(x))e^k.$$

If $g \in C_A^*$, then

$$g(x) = h(x)\, g(e) + \sum_k (x_k - h(x))\, g(e^k).$$

From the form (3) of a member of C_A^*, we see that $\Sigma_k g(e^k)$ converges, hence

$$g(x) = h(x)\left(g(e) - \sum_k g(e^k)\right) + \sum_k x_k g(e^k).$$

Thus C_A has FAK, and by Lemma 3, A is associative. If B is a matrix such that $C_B = C_A$, then $\lim_n B_n(x)$ is a continuous linear functional on C_A. Since $\lim_n B_n(e^k) = b_k$, A has PMI, by Lemma 3.

LEMMA 5. *If* A *is a replaceable associative matrix, then* Φ *is a basis for* C_A.

Proof. There exists a regular matrix B such that $C_B = C_A$. Let

$$\lim_n B_n(x) = h(x)$$

for $x \in C_A$. Then

$$h(x) = t \lim_n A_n(x) + \sum_n t_n A_n(x) + \sum_k \beta_k x_k ,$$

where $\{t_n\} \in (\gamma)$. By [3, Lemma 8], A is a co-regular, and since

$$1 = h(e) - \sum_k h(e^k) = t\left(\lim \sum_k a_{nk} - \sum_k a_k\right),$$

we see that $t \neq 0$. Therefore, by [4, Theorem 5.3], we may assume that $B = (b_{nk})$, where

$$b_{0k} = \beta_k ,$$

$$b_{1k} = \beta_k + ta_{0k} \qquad (k = 0, 1, 2, \cdots),$$

$$b_{nk} = \beta_k + \sum_{j=0}^{n-2} t_j a_{jk} + ta_{n-1,k} \qquad (n \geq 2).$$

Suppose $\{c_n\} \in (\gamma)$. Then

$$\sum_n c_n b_{nk} = \beta_k \sum_n c_n + t \sum_n c_{n+1} a_{nk} + \sum_{n=2} c_n \sum_{j=0}^{c-2} t_j a_{jk} ,$$

so that $\Sigma_k \Sigma_n c_n b_{nk} x_k$ converges for all $x \in C_A$ provided $\Sigma_k [\Sigma_{n=2}^{\infty} c_n \Sigma_{j=0}^{n-2} t_j a_{jk}] x_k$ does. Now

$$\sum_{n=2}^{\infty} c_n \sum_{j=0}^{n-2} t_j a_{jk} = \sum_j \sum_{n=j}^{\infty} c_{n+2} t_j a_{jk} ,$$

since either side is absolutely convergent. Since $\Sigma_j |\Sigma_{n=j}^{\infty} c_{n+2} t_j|$ converges and A is associative, $\Sigma_k (\Sigma_j \Sigma_{n=j}^{\infty} c_{n+2} t_j a_{jk}) x_k$ converges for all $x \in C_A$. Thus

$$\sum_k \sum_n c_n b_{nk} x_k$$

converges for all $x \in C_A$. Since $C_A = C_B$, B is associative. But $b_k = 0$ $(k = 0, 1, 2, \cdots)$, so that by Lemma 3, C_B has FAK. The set N_B of all x such that $\lim_n B_n(x) = 0$ is a closed linear subspace of C_B, and if $f \in N_B^*$, f may be extended to a continuous linear functional on C_B (see [4]). Thus N_B has FAK. By Lemma 1, $\overline{H(\Phi)} = C_B$, so that $\overline{H(\Delta)} = N_B$, since B is regular. Therefore, by [5, Theorem 3.4], Δ is a basis for N_B. If $x \in C_B$, then $x = h(x)e + y$ with $y \in N_B$. Hence

$$y = \sum_k y_k e^k = \sum_k (x_k - h(x))\, e^k \qquad \text{and} \qquad x = h(x)e + \sum_k (x_k - h(x))\, e^k .$$

It is easily seen that the coefficients of e and e^k ($k = 0, 1, 2, \cdots$), are unique. Thus Φ is a basis for C_B, and since C_B and C_A have the same topology [4, Theorem 4.5], Φ is a basis for C_A.

The following lemma is a restatement of [3, Lemma 14 and Theorem 3].

LEMMA 6. *Suppose* A *is conservative and has* PMI. *Then* A *is associative. If in addition* A *is co-regular, then* A *is replaceable.*

From Lemmas 4, 5 and 6 we have the following two theorems.

THEOREM 1. *Let* A *be a replaceable matrix. The following are equivalent:*

(8) Φ *is a basis for* C_A.

(9) A *is associative.*

(10) A *has propagation of maximal inset.*

THEOREM 2. *Let* A *be a co-regular matrix. Then* Φ *is a basis for* C_A *if and only if* A *has propagation of maximal inset.*

THEOREM 3. *Let* A *be a conservative matrix.* Δ *is a basis for* C_A *if and only if*

(11) A *is associative, and*

(12) $$\lim_n A_n(x) = \sum_k a_k x_k \quad \textit{for all } x \in C_A .$$

Proof. If Δ is a basis for C_A, then $x = \Sigma_k x_k e^k$ for $x \in C_A$. If $\{t_n\} \in (\gamma)$, let $f(x) = \Sigma_n t_n A_n(x)$. Then $f \in C_A^*$, and it follows that

$$\sum_n t_n \sum_k a_{nk} x_k = \sum_k \sum_n t_n a_{nk} x_k \qquad (x \in C_A),$$

so that A is associative. (12) is obviously true. Conversely, if (11) and (12) hold, let $f \in C_A^*$. Then

$$f(x) = t \lim_n A_n(x) + \sum_n t_n A_n(x) + \sum_k \beta_k x_k$$

$$= t \sum_k a_k x_k + \sum_k \sum_n t_n a_{nk} x_k + \sum_k \beta_k x_k = \sum_k x_k f(e^k)$$

for each $x \in C_A$. By [5, Theorem 3.4], Δ is a basis for C_A. Clearly, if Δ is a basis for C_A, then A has PMI. Further, if $C_B = C_A$, then $\lim_n B_n(x) = \Sigma_k b_k x_k$ for all $x \in C_A$.

Note that wherever the equation $(AB)x = A(Bx)$ has occurred, the restriction that $\|B\|$ be finite may be replaced by the condition that either $\Sigma_n |b_{nk}|$ be convergent for each n, or that B be conservative. If $\{b_n\} \in (\gamma)$, the elements b_n may be made the elements of a row of a matrix of any of the above types.

A conservative matrix A may be associative while Φ fails to be a basis for C_A; this can be seen from the following example. Let $A = (a_{nk})$, where $a_{nk} = c_n b_k$ with $\Sigma_k |b_k|$ convergent and $\lim_n c_n = 0$. If $\{t_n\} \in (\gamma)$ and $x \in C_A$, then

$$\sum_k \sum_n t_n a_{nk} x_k = \sum_k \sum_n t_n c_n b_k x_k,$$

and therefore the double series on the left converges. Obviously, A satisfies (12). Thus Δ is a basis for C_A (compare [3, Theorem 2]).

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