A SIMPLIFIED PROOF OF WARING'S CONJECTURE

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The purpose of this paper is to give a short-cut in the proof of Waring's conjecture. The novelty in our procedure lies in the use of some of the elementary number theoretic notions due to Schnirelmann, which allow us to employ crude upper bounds in the circle method, rather than the usual asymptotic formulae.

Our starting point is the estimate for the "Weyl sums" (see [3, Volume I, Part 2, p. 255]), a special form of which we state as our

LEMMA 1. *Let* $k > 1$ *be a fixed integer. There exists a* $\delta > 0$ *and a* C_1 *such that, for any positive integers* N, a, b *with* $(a, b) = 1$ *and* $N^{1/2} \leq b \leq N^{k-1/2}$,

$$\left| \sum_{n=1}^{N} e\left(\frac{a}{b} n^k\right) \right| \leq C_1 N^{1-\delta}.$$

(Throughout, we write $e(t) = e^{2\pi i t}$, and $C_1, C_2, \cdots$ denote constants.) Our end-point will be the

THEOREM. *If, for each positive integer* s, *we write*

$$r_s(n) = \sum_{\substack{n_1^k + \cdots n_s^k = n \\ n_i \geq 0}} 1,$$

then there exist g *and* C *such that* $r_g(n) \leq C n^{g/k-1}$ *for all* $n > 0$.

The previously cited notions of Schnirelmann allow the deduction, from this theorem, of the full Waring result, namely:

There exists a G *for which* $r_G(n) > 0$ *for all* $n > 0$. For the details see [1, pp. 40, 41].

To prove our theorem: since

$$r_g(n) = \int_0^1 \left(\sum_{m \leq n^{1/k}} e(xm^k) \right)^g e(-nx)\, dx,$$

it suffices to prove that there exist g and C for which

$$(1) \qquad \int_0^1 \left| \sum_{n=1}^{N} e(xn^k) \right|^g dx \leq C N^{g-k} \qquad \text{for all } N > 0.$$

First some parenthetical remarks about this inequality: Suppose it is known to hold for some C_0 and g_0; then, since $\left| \sum_{n=1}^{N} e(xn^k) \right| \leq N$, it persists for C_0 and any

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$g \geq g_0$. Thus (1) is a property of large g's, in other words, it is purely a "magnitude property." Again, (1) is a best possible inequality in that, for each g, there exists a c such that

$$\int_0^1 \left| \sum_{n=1}^{N} e(xn^k) \right|^g dx > c\, N^{g-k} \quad \text{for all } N. \tag{2}$$

To see this, note that $\sum_{n=1}^{N} e(xn^k)$ has a derivative bounded by $2\pi N^{k+1}$. Hence, in the interval $\left(0, \frac{1}{4\pi N^k}\right)$,

$$\left| \sum_{n=1}^{N} e(xn^k) \right| \geq N - 2\pi N^{k+1} \cdot \frac{1}{4\pi N^k} = \frac{N}{2},$$

and so (2) follows with $c = \frac{1}{4\pi \cdot 2^g}$.

The remainder of our paper, then, will be devoted to the derivation of (1) from Lemma 1.

Denote by $I_{a,b,j,N}$ the interval defined by

$$\left| x - \frac{a}{b} \right| \leq \frac{1}{b N^{k-1/2}}, \quad 2j - 1 \leq N^k \left(x - \frac{a}{b} \right) \leq 2j + 1,$$

where a, b, j, N are integers such that $N > 0$, $b > 0$, $0 \leq a < b$, $(a, b) = 1$, $b \leq N^{k-1/2}$. By Dirichlet's theorem, these intervals cover $(0, 1)$ for each fixed N. Our main tool is

LEMMA 2. *There exist* $\varepsilon > 0$ *and* C_2 *such that, throughout any interval* $I_{a,b,j,N}$,

$$\left| \sum_{n=1}^{N} e(xn^k) \right| \leq \frac{C_2 N}{(b + |j|)^{\varepsilon}}.$$

Proof. This is almost trivial if $b > N^{2/3}$, for since the derivative of $\sum_{n=1}^{N} e(xn^k)$ is bounded by $2\pi N^{k+1}$, we have

$$\left| \sum_{n=1}^{N} e(xn^k) \right| \leq \left| \sum_{n=1}^{N} e\left(\frac{a}{b} n^k\right) \right| + \left| x - \frac{a}{b} \right| \cdot 2\pi N^{k+1}$$

$$\leq \frac{C_1 N}{b^{\delta/k}} + \frac{2\pi N^{2/3}}{b} \qquad \text{(by Lemma 1)}$$

$$\leq \frac{C_1 N}{b^{\delta/k}} + \frac{2\pi N}{b^{1/4}},$$

which gives the result, since $j = 0$ automatically. Assume therefore that $b \leq N^{2/3}$, and note the following two simple facts (for details see [2, p. 313] and [4, Vol. I, Part II, p. 37], respectively).

(A) If M is the maximum of the moduli of the partial sums $\sum_{n=1}^{m} a_n$, V the total variation of $f(t)$ in $0 \leq t \leq N$, and M' the maximum of the modulus of $f(t)$ in $0 \leq t \leq N$, then

$$\left| \sum_{n=1}^{N} a_n f(n) \right| \leq M(V + M').$$

(B) If V is the total variation of $f(t)$ in $0 \leq t \leq N$, then

$$\left| \sum_{n=1}^{N} f(n) - \int_0^N f(t)\, dt \right| \leq V .$$

Now write $\alpha = \frac{1}{b} \sum_{n=1}^{b} e\left(\frac{a}{b} n^k\right)$ and

(3) $$\sum_{n=1}^{N} e(xn^k) = S_1 + \alpha S_2 ,$$

where

$$S_1 = \sum_{n=1}^{N} \left\{ e\left(\frac{a}{b} n^k\right) - \alpha \right\} e\left(\left(x - \frac{a}{b}\right) n^k \right), \quad S_2 = \sum_{n=1}^{N} e\left(\left(x - \frac{a}{b}\right) n^k \right) .$$

We apply (A) to S_1; to do so, we note that

$$\left| \sum_{n=1}^{m} \left\{ e\left(\frac{a}{b} n^k\right) - \alpha \right\} \right| = \left| 0 + \sum_{b[m/b] < n \leq m} \left\{ e\left(\frac{a}{b} n^k\right) - \alpha \right\} \right| \leq (1 + |\alpha|)\, b \leq 2b .$$

Also, the total variation of $e\left(\left(x - \frac{a}{b}\right) t^k \right)$ is equal to $2\pi \left| x - \frac{a}{b} \right| N^k \leq \frac{2\pi \sqrt{N}}{b}$, while $M' = 1$. The result is

(4) $$|S_1| \leq 4\pi \sqrt{N} + 2b \leq 5\pi N^{2/3}.$$

Next we apply (B) to S_2 and obtain

(5) $$|S_2| \leq \left| \int_0^N e\left(\left(x - \frac{a}{b}\right) t^k \right) dt \right| + \frac{2\pi \sqrt{N}}{b}$$

and thus, calling $J = N^k \left| x - \frac{a}{b} \right|$, and noting that $J \geq |j| - 1$, we have

$$\left|\int_0^N e\left(\left(x-\frac{a}{b}\right)t^k\right)dt\right| = \frac{N}{J^{1/k}}\left|\int_0^{J^{1/k}} e(u^k)\,du\right| \le \frac{NC_3}{J^{1/k}},$$

since $\int_0^\infty e(u^k)\,du$ converges. Since the above integral is trivially bounded by N, we can write

$$\left|\int_0^N e\left(\left(x-\frac{a}{b}\right)t^k\right)dt\right| \le \frac{C_4 N}{(2+J)^{1/k}} \le \frac{C_4 N}{(1+|j|)^{1/k}}.$$

Combining this with (5) gives

$$|\alpha S_2| \le \frac{C_4 N|\alpha|}{(1+|j|)^{1/k}} + 2\pi\sqrt{N}. \tag{6}$$

If we now apply Lemma 1 to the case $N = b$, we obtain $|\alpha| \le C_1/b^\delta$, and by (3), the addition of (4) and (6) gives

$$\left|\sum_{n=1}^N e(xn^k)\right| \le \frac{C_5 N}{b^\delta(1+|j|)^{1/k}} + 7\pi N^{2/3} \le \frac{C_5 N}{b^\delta(1+|j|)^{1/k}} + \frac{C_6 N}{(b+|j|)^{1/2}}.$$

Since $j \le \sqrt{N}$, $b \le N^{2/3}$. The choice

$$C_2 = C_5 + C_6 + C_1 + 2\pi,\quad \varepsilon = \min\left(\frac{\delta}{k}, \frac{1}{k}, \frac{1}{4}\right)$$

completes the proof.

Proof of (1). Choose $g \ge 4/\varepsilon$, ε given by Lemma 2. By Lemma 2, since the length of $I_{a,b,j,N}$ is at most $2N^{-k}$,

$$\int_{I_{a,b,j,N}} \left|\sum_{n=1}^N e(xn^k)\right|^g dx \le \frac{C_7 N^g}{(b+|j|)^4}\cdot\frac{1}{N^k}.$$

Summing over all a, b, j gives the estimate

$$C_7 N^{g-k}\sum_{b,j}\frac{b}{(b+|j|)^4} \le C N^{g-k},$$

since $\sum_{b=1}^\infty \sum_{j=-\infty}^\infty \frac{1}{(b+|j|)^3} < \infty$. This completes the proof, since the intervals $I_{a,b,j,N}$ cover (0, 1).

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