# ON POLYHEDRA IN SPACES OF CONSTANT CURVATURE

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An elementary procedure for proving the formula

$$F = \alpha + \beta + \gamma - \pi, \tag{1}$$

where F is the area of a geodesic triangle on the unit sphere and where $\alpha, \beta, \gamma$ are the angles, leads to analogous relations for polyhedra in spaces of constant positive curvature of an even number of dimensions. We shall here confine ourselves to the case of the four-dimensional spherical space $\mathfrak{S}_4$ of curvature $+1$.

## 1. PENTAHEDRA

We shall first derive an analogue to (1) which expresses the volume of a pentahedron in terms of the solid angles at its vertices and the angles formed by the lateral surfaces.

The five lateral surfaces are tetrahedra in three-dimensional spaces of curvature $+1$. Each of these spaces divides $\mathfrak{S}_4$ into two congruent semispaces which shall be denoted by $1, \bar{1}; 2, \bar{2}; 3, \bar{3}; 4, \bar{4}; 5, \bar{5}$, corresponding to the five lateral surfaces $S_1, S_2, S_3, S_4, S_5$. The set of the interior points of the pentahedron can be assumed to be the intersection of the semispaces 1, 2, 3, 4, 5. Let V be the volume of the pentahedron, and let (1, 2, 3, 4, 5) be the volume of the intersection of 1, 2, 3, 4, 5. We then have

$$V = (1, 2, 3, 4, 5). \tag{2}$$

It is obvious that this notation immediately furnishes the relations

$$(1, 2, 3, 4, 5) + (1, 2, 3, 4, \bar{5}) = (1, 2, 3, 4), \tag{3}$$

where (1, 2, 3, 4) is the intersection of the spaces 1, 2, 3, 4. Taking into account that

$$(1, 2, 3, 4, 5) = (\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}), \tag{4}$$

we obtain the formulae

$$5V + \sum (1, 2, 3, 4, \bar{5}) = \sum (1, 2, 3, 4), \tag{5a}$$

$$4\sum (1, 2, 3, 4, \bar{5}) + 2\sum (1, 2, 3, \bar{4}, \bar{5}) = \sum (1, 2, 3, \bar{5}), \tag{5b}$$

$$V + \sum (1, 2, 3, 4, \bar{5}) + \sum (1, 2, 3, \bar{4}, \bar{5}) = \frac{4\pi^2}{3}, \tag{5c}$$

where the summations extend over all possible distributions of the bars among the numbers 1, 2, 3, 4, 5.

Received January 25, 1960.

The formula (5c) expresses the fact that the sum of the volumes of all the 32 possible intersections of the semispaces $1, \bar{1}; \cdots; 5, \bar{5}$ equals the volume of $\mathfrak{S}_4$. The equations (5a), (5b), (5c) enable us to express V in terms of the volumes of the intersections of only four semispaces:

$$V = \frac{2}{9}\pi^2 + \frac{1}{6}\sum(1, 2, 3, 4) - \frac{1}{12}\sum(1, 2, 3, \bar{4}). \tag{6}$$

Let us now discuss the meaning of (1, 2, 3, 4). This expression is closely related to the solid angle of the pentahedron at that vertex where the lateral surfaces $S_1$, $S_2$, $S_3$, $S_4$ intersect each other. The intersection of the polar plane $\mathfrak{S}_3$ (three-dimensional space $\mathfrak{S}_3$ of curvature 1) of this vertex with the three-dimensional planes determined by $S_1$, $S_2$, $S_3$, $S_4$ forms a tetrahedron whose volume $\omega_5$ in $\mathfrak{S}_3$ is the solid angle at that vertex. Since the volume of $\mathfrak{S}_3$ equals $2\pi^2$, we have

$$\omega_5 = \frac{2\pi^2}{8\pi^2/3}(1, 2, 3, 4), \qquad (1, 2, 3, 4) = \frac{4}{3}\omega_5. \tag{7}$$

Equations (7) give the relation between (1, 2, 3, 4) and the solid angle $\omega_5$.

We shall now deal with the second sum in equation (6). It is evident that

$$4\sum(1, 2, 3, 4) + \sum(1, 2, 3, \bar{4}) = 2\sum(1, 2, 3), \tag{8}$$

where (1, 2, 3) is the volume of the intersection of the semispaces 1, 2, 3. The polar space of this intersection is a two-dimensional spherical space. A spherical triangle is cut out of this space by the planes corresponding to $S_1$, $S_2$, $S_3$. Let $\omega_{45}$ be its area. We then have

$$(1, 2, 3) = \frac{\omega_{45}}{4\pi}\cdot\frac{8\pi^2}{3} = \frac{2\pi}{3}\omega_{45} \tag{9}$$

or, according to the classical formula (1),

$$(1, 2, 3) = \frac{2\pi}{3}(\alpha_{12} + \alpha_{13} + \alpha_{23} - \pi), \tag{10}$$

where $\alpha_{ik}$ denotes the angle included by the lateral surfaces $S_i$, $S_k$. From equation (10) we conclude that

$$\sum(1, 2, 3) = 2\pi\sum\alpha_{ik} - \frac{20\pi^2}{3}. \tag{11}$$

The combination of equations (6), (7), (8), (11) yields the fundamental equation

$$\boxed{V = \frac{4}{3}\pi^2 + \frac{2}{3}\sum\omega_i - \frac{\pi}{3}\sum\alpha_{ik},} \tag{12}$$

*which expresses the volume of a pentahedron in $\mathfrak{S}_4$ in terms of the solid angles at its vertices and the angles between the lateral surfaces.* Equation (12) is the analogue to the classical formula (1).

## 2. CONVEX POLYHEDRA

Formula (12) will now be generalized for convex polyhedra in $\mathfrak{S}_4$ with an arbitrary number of lateral surfaces. "Convex" means that the polyhedron is the intersection of semispaces as mentioned above. The boundary of the polyhedron consists then of

(1) convex polyhedra $S_1^{(3)}, \cdots, S_v^{(3)}$ in three-dimensional spherical spaces;

(2) convex polygons $S_1^{(2)}, \cdots, S_f^{(2)}$ on unit spheres;

(3) great circles $S_1^{(1)}, \cdots, S_k^{(1)}$; and

(4) vertices $S_1^{(0)}, \cdots, S_e^{(0)}$.

The number of lateral surfaces, edges, and vertices of an $S_i^{(3)}$ may be denoted by $f_v$, $k_v$, $e_v$, respectively; the number of sides or vertices of an $S_k^{(2)}$ by $k_f$. We now decompose the polyhedron into elementary pentahedra in the following way:

1. In the interior of the polyhedron we choose an arbitrary point M.
2. In the interior of each $S_i^{(3)}$ we choose a point $M_i'$.
3. In the interior of each $S_i^{(2)}$ we choose a point $M_i''$.
4. We take two neighboring vertices A and B of $S_i^{(2)}$.

The points M, $M_i'$, $M_i''$, A, B are the vertices of an elementary pentahedron, the volume of which can be determined by means of formula (12). In order to find the volume of the polyhedron, we have to sum over all elementary pentahedra. The sum can be evaluated as follows: We consider the ten two-dimensional lateral surfaces (geodesic triangles) of an elementary pentahedron, for instance A, M, $M_i'$, and sum the expression (12) over all the elementary pentahedra which have this two-dimensional lateral surface in common. Let us first investigate how much the second term in (12) contributes to the volume of the polyhedron. It is obvious that the sum of all solid angles at M equals the volume of a three-dimensional spherical space, that is, $2\pi^2$. The sum of the solid angles at $M_i'$, however, equals $\pi^2$, that is, the volume of a three-dimensional semispace. At $M_i''$ the sum of the solid angles corresponds to that part of the three-dimensional polar plane of $M_i''$ which is cut out by those spaces $S_2^{(3)}$, $S_n^{(3)}$ which intersect each other in $S_i^{(2)}$. The contribution of the solid angles at $M_i''$ is therefore

$$\frac{2\pi^2}{2\pi}\alpha_i = \pi\alpha_i,$$

where $\alpha_i$ is the angle ($< \pi$) included by $S_m^{(3)}$ and $S_n^{(3)}$, that is, the angle included by the three-dimensional lateral surfaces which intersect each other in $S_i^{(2)}$.

At the vertices A and B of the polyhedron, the sum of the solid angles of the elementary pentahedra is identical with the solid angles $\psi_i$ of the polyhedron at these vertices.

Our considerations finally enable us to extend the sum in the second term of equation (12) over the whole polyhedron, and to express this sum in terms of the solid angles $\psi_i$ of the polyhedron at its vertices and the angles $\alpha_i$ included by the three-dimensional lateral surfaces:

$$\sum \omega_i = 2\pi^2 + \pi^2 v + \pi \sum_f \alpha_i + \sum_e \psi_i, \tag{13}$$

where e, f, v are the numbers of vertices and of two-dimensional and three-dimensional surfaces, respectively.

Let us now deal with the third term in formula (12). In order to extend the sum $\Sigma \alpha_{ik}$ over all elementary pentahedra, we list their 10 two-dimensional lateral surfaces, and we list for each of them the contribution made by those pentahedra which have in common a lateral two-dimensional surface of the same type.

| Type of Lateral Surface | Contribution of the Adjacent Pentahedra |
|---|---|
| M M' M" | $2\pi \sum_v f_v$ |
| A B M' | $\pi \sum_v k_v$ |
| A B M" | $\sum_f \alpha_f k_f$ |
| A M' M; B M' M | $2\pi \sum_v e_v$ |
| A B M | $2\pi k$ |
| A M" M; B M" M | $2\pi \sum_f k_f = 2\pi \sum_v k_v$ |
| A M' M"; B M' M" | $2\pi \sum_v k_v$ |

Here $\Sigma_v$ and $\Sigma_f$ mean that the summation extends over all three-dimensional, and two-dimensional lateral surfaces, respectively. Taking advantage of Euler's formula

$$e_v + f_v - k_v = 2,$$

we obtain

(14) $\sum \alpha_{ik}$ (extended over all pentahedra) $= 4\pi v + 7\pi \sum_v k_v + 2\pi k + \sum_f \alpha_f k_f$.

The first term $\frac{4\pi^2}{3}$ contributes

(15) $$\frac{8\pi^2}{3} \sum_v k_v .$$

Combining the expressions (13), (14), and (15), and taking into consideration the equations

(16) $$\sum_v k_v = \sum_f k_f$$

and

(17) $$e - k + f - v = 0,$$

we obtain the final result:

(18) $$V = \frac{4\pi^2}{3} + \frac{\pi}{3} \sum_f (\pi - \alpha_f)(k_f - 2) - \frac{2}{3} \sum_e (\pi^2 - \psi_i).$$

*Equation* (18) *expresses the volume of a polyhedron in terms of the angles* $\alpha_f$ *included by the three-dimensional lateral surfaces and the solid angles at the vertices.*

It is interesting to make the transition from non-Euclidean to Euclidean space. From equation (18) we immediately obtain the following formula, valid for any convex body in four-dimensional Euclidean space:

(19) $$0 = \frac{4\pi^2}{3} + \frac{\pi}{3} \sum_f (\pi - \alpha_f)(k_f - 2) - \frac{2}{3} \sum_e (\pi^2 - \psi_i).$$

Equations (18) and (19) seem to have far-reaching consequences. Equation (18), for instance, suggests the conjecture that the volume of a convex body in a four-dimensional non-Euclidean space depends only on the $g_{ik}$ of its boundary.

It is appealing to verify equation (19) for the case where the polyhedron is a cube in four-dimensional Euclidean space. Here we have

$$\alpha_f = \frac{\pi}{2}, \quad k_f = 4, \quad f = 24, \quad e = 16, \quad \psi_i = \frac{\pi^2}{8},$$

and equation (19) becomes an identity.

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