GROUP COMMUTATORS OF BOUNDED OPERATORS IN HILBERT SPACE

C. R. Putnam

1. In the sequel only bounded (linear) operators A, B, $\cdots$ on a Hilbert space of elements x, y, $\cdots$ will be considered. For any such operator A, let $W = W(A)$ denote the closure of the set of complex numbers (Ax, x) with $\|x\| = (x, x)^{1/2} = 1$. It is known that W is a bounded convex set containing sp(A), the spectrum of A, and that in case A is normal, W is the least closed convex set containing sp(A) (Hausdorff, Toeplitz); see, for example [4], pp. 34 ff. An operator A will be called *nonsingular (invertible)* if it possesses a unique right (hence, a unique left) bounded inverse A^{-1}. In case A and B are nonsingular, let $D = ABA^{-1}B^{-1}$, the group commutator of A and B. It will be supposed throughout this paper that A commutes with D, so that

$$(1) \qquad AD = DA, \qquad D = ABA^{-1}B^{-1}.$$

It is known that if, in addition to (1), A and B are finite-dimensional unitary matrices and if the spectrum of B is contained in some open semicircle on the circle $|z| = 1$, then necessarily $D = I$, that is, $AB = BA$; see [2, Theorem 197], also [3]. In the present paper various generalizations of this result will be obtained; in particular it will be shown that the restriction that A and B be finite matrices can be removed. Since, when B is unitary, the above assumption concerning sp(B) is equivalent to the condition that 0 fails to belong to the set W(B), it is clear that the earlier assertion for the case where A and B are finite-dimensional and unitary is contained in

(I) *Let* A *and* B *be unitary (so that* $D = ABA^{-1}B^{-1}$ *is unitary) and satisfy* (1). *Then either* $AB = BA$ *or* 0 *belongs to the set* W(B).

If N is any nonsingular normal operator, it is easy to see that 0 belongs to the set W(N) if and only if 0 belongs to the set $W(N^{-1})$. Consequently, (I) is seen to be a consequence of the more general result

(II) *Let* A *be unitary, and let* B *be an arbitrary nonsingular operator satisfying* (1). *Then at least one of the following cases must hold:* (i) sp(D) = 1 *only, or* (ii) 0 *belongs to* W(B), *or* (iii) 0 *belongs to* $W(B^{-1})$.

2. *Proof of* (II). Suppose that z belongs to sp(D); then, as $m \to \infty$,

$$(2) \qquad \text{either} \quad (D - z)x_m \to 0 \quad \text{or} \quad (D^* - \bar{z})x_m \to 0,$$

for some sequence of elements x_m satisfying $\|x_m\| = 1$. It will be shown that if $z \neq 1$, that is, if (i) fails to hold, condition (2) implies that either (ii) or (iii) of (II) must hold.

It is seen from $DB = ABA^{-1}$ and an application of (1) that $D^2B = A^2BA^{-2}$ and, in general, that

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(3) $$D^nB = A^nBA^{-n} \quad \text{for } n = 0, 1, 2, \cdots.$$

Condition (1) implies $ABA^{-1}B^{-1} = BA^{-1}B^{-1}A$ and hence $B^{-1}D = A^{-1}B^{-1}A$ and, as above,

(4) $$B^{-1}D^n = A^{-n}B^{-1}A^n \quad \text{for } n = 0, 1, 2, \cdots.$$

Next, suppose the first alternative of (2) holds. Then it follows that $(D^n - z^n)x_m \to 0$ as $m \to \infty$, for every fixed $n = 0, 1, 2, \cdots$. Hence

(5) $$(B^{-1}D^n x_m, x_m) = z^n(B^{-1}x_m, x_m) + \alpha_{nm},$$

where $\alpha_{nm} \to 0$ as $m \to \infty$ for n fixed.

Since A (hence A^n) is unitary, it follows from (4) that $W(B^{-1}D^n) = W(B^{-1})$ for every n. Since $W(B^{-1})$ is convex, the center of mass $M^{-1}\Sigma_{k=1}^{M} y_k$ of any M points $y_1, \cdots, y_M$ in $W(B^{-1})$ also belongs to $W(B^{-1})$. It now follows from (5) that c_{Mm}, where

(6) $$c_{Mm} = (B^{-1}x_m, x_m)(z - z^{M+1})/M(1 - z) + M^{-1}\sum_{n=1}^{M}\alpha_{nm},$$

belongs to $W(B^{-1})$.

Next, if $|z| \leq 1$, it is clear that the first term on the right of (6) tends to zero as $M \to \infty$ (uniformly in m). Hence, given $\varepsilon > 0$, one can choose $M = M_\varepsilon$ so large that the absolute value of the first term is less than ε. Since, for $M = M_\varepsilon$ fixed, the second term tends to 0 as $m \to \infty$, it is clear that $\liminf_{M,m\to\infty} |c_{Mm}| = 0$. Since $W(B^{-1})$ is closed, this means that 0 belongs to $W(B^{-1})$, that is, (iii) of (II) holds.

Next, suppose that $|z| > 1$, If $b = \liminf_{m\to\infty} |(B^{-1}x_m, x_m)|$ satisfies $b = 0$, then again (iii) holds. Thus, it can be supposed that $b > 0$. It is clear that for $\varepsilon > 0$, $M = M_\varepsilon$ can be chosen so large that the absolute value of the first term on the right of (6) exceeds $1/\varepsilon$, for all large m, while for $M = M_\varepsilon$ fixed, the second term tends to 0 as $m \to \infty$. Hence $\limsup_{M,m\to\infty} |c_{Mm}| = \infty$ and so $W(B^{-1})$ is unbounded, hence B^{-1} is unbounded, a contradiction.

Thus the proof of (I) is complete if the first alternative of (2) holds. In case the second possibility holds, one need only use (3) instead of (4) and proceed in a similar manner to conclude that now 0 belongs to $W(B)$. This completes the proof of (II).

3. The proof of (II) implies

(III) *Let* A *and* B *satisfy the same assumptions as in* (II). *In addition, suppose that whenever* z *is in* $\text{sp}(D)$ *with* $z \neq 1$, *then both* $(D - z)x_m \to 0$ *and* $(D^* - \bar{z})y_m \to 0$ *as* $m \to \infty$, *for some pair of sequences* $\{x_m\}$ *and* $\{y_m\}$ *of unit elements. Then either* $\text{sp}(D) = 1$ *only, or* 0 *belongs both to* $W(B)$ *and to* $W(B^{-1})$.

As an immediate consequence of (III) and the remark following (I), one obtains

(IV) *Let* A *and* B *satisfy the same assumptions as in* (II) *and, in addition, suppose that* D *is normal. Then either* $AB = BA$ *or* 0 *belongs both to* $W(B)$ *and to* $W(B^{-1})$.

In the next section, some similar theorems with modified hypotheses will be obtained.

4. (V) *Suppose that* A *and* B *are nonsingular operators satisfying* (1), *and that either*

(7) $$D^n B \text{ is normal for } n = 0, 1, 2, \cdots,$$

or

(7') $$D^{-n} B \text{ is normal for } n = 0, 1, 2, \cdots.$$

(Either (7) *or* (7') *implies, in particular, that* B *is normal.) Then either* sp(D) = 1 *only, or* 0 *belongs both to* W(B) *and to* $W(B^{-1})$.

Proof of (V). First suppose that (7) holds. Then according to (3), $D^n B$ is equivalent to B and, since $D^n B$ and B are normal, $D^n B$ is unitarily equivalent to B; see [1]. Hence $W(D^n B) = W(B)$. If z belongs to sp(D) then (2) holds, and if sp(D) does not consist of 1 only, then it can be supposed that $z \neq 1$. In case the second alternative of (2) holds, then the proof proceeds, as that of (I), in order to show that 0 is in W(B). If the first alternative of (2) holds, then $B^{-1} D^{-n} = (D^n B)^{-1}$ is normal and moreover, on taking inverses in (3), one has

(8) $$B^{-1} D^{-n} = A^n B^{-1} A^{-n} \quad \text{for } n = 0, 1, 2, \cdots.$$

As above, $B^{-1} D^{-n}$ is unitarily equivalent to B^{-1}, and the proof proceeds to show that 0 is in $W(B^{-1})$ as in the first part of the proof of (I), but using (8) rather than (4) as there. (It is clear that (8) could also have been used in Section 2.) Since condition (7) implies that B is normal, it follows from the remark after (I) that 0 belongs both to W(B) and to $W(B^{-1})$, and the proof of (V) is complete in case (7) holds. If (7') holds, the argument is similar, and the proof of (V) is now complete.

A corollary of (V) is

(VI) *Let* A *be arbitrary,* B *and* D *normal, all nonsingular, and suppose that* (1) *holds. In addition, suppose that* D *commutes with* BB^* *and that* B *commutes with* DD^*. *Then either* AB = BA *or* 0 *belongs both to* W(B) *and to* $W(B^{-1})$.

5. In this section, two further theorems will be given.

(VII) *Let* A *and* B *be arbitrary nonsingular operators satisfying* (1). *Suppose that* D *possesses a square root* $D^{1/2}$ *such that* $(D^{1/2})^n B (D^{1/2})^n$ *is normal for* $n = 0, 1, 2, \cdots$. *(In particular,* B *is normal.) Then either* sp(D) = 1 *only, or* 0 *belongs both to* W(B) *and to* $W(B^{-1})$.

The proof is similar to that of (V), if it is noted that

$$D^n B = D^{n/2}(D^{n/2} B D^{n/2}) D^{-n/2}$$

and hence, by (3), $D^{n/2} B D^{n/2}$ is equivalent, hence unitarily equivalent, to B.

A corollary of (VII) is

(VIII) *Let* A *be arbitrary,* B *and* D *self-adjoint,* D *positive, and suppose that* (1) *holds. Then either* AB = BA, *or* 0 *belongs both to* W(B) *and to* $W(B^{-1})$.

The proof follows if one chooses $D^{1/2}$ to be the (unique) positive square root of D and notes that $D^{n/2} B D^{n/2}$ is then self-adjoint for $n = 0, 1, 2, \cdots$.

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Purdue University.