PARTITIONS INTO PRIME POWERS

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1. INTRODUCTION

Let p(n, m; k) stand for the number of partitions of the integer n into k-th powers of primes p $(2 \leq p \leq m)$. Clearly, it is sufficient to consider only $m \leq n^{1/k}$. If $m \geq n^{1/k}$, or if k = 1, mention of the corresponding parameter is usually omitted, and we write simply p(n; k), or p(n, m); the total number of partitions of n into primes is denoted by p(n). Hardy and Ramanujan [4] proved that

$$\log p(n, k) \sim (k+1)\left\{\Gamma\left(2+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right\}^{k/(k+1)}\left(\frac{n}{\log^k n}\right)^{1/(k+1)},$$

so that, in particular,

$$\log p(n) \sim 2\pi\left(\frac{n}{3 \log n}\right)^{1/2}.$$

Brigham [2] obtained an asymptotic formula for a certain weighted partition function. In 1953, Haselgrove and Temperley [5] obtained an asymptotic formula for partitions into parts (with or without restriction on their number) selected from some preassigned set A of integers, provided that A satisfies certain conditions. The results obtained are of considerable generality, because most sets of interest satisfy the required conditions almost trivially; the primes, however, are a borderline case. The formulae still hold, but the corresponding justification is far from simple, and it is presented (as are some other points of the paper) rather sketchily. Actually, Haselgrove and Temperley's formula is valid even for the partitions into prime powers p(n; k), but that is neither justified, nor even claimed in the paper. This may account for the fact that Mitsui, who in 1957 obtained [8] an asymptotic formula for p(n, m; k), credits Haselgrove and Temperley only with the determination of p(n, n; 1) instead of p(n, n; k). The more general formulae of Haselgrove and Temperley and of Mitsui are not directly comparable, because the former's restrictions refer to the number of parts, while the latter's refer to the size of the summands admitted. While preceding papers make use of the theory of functions of complex variables, Bateman and Erdös [1] use a rather elementary approach, in order to prove that p(n, m) is an increasing function of both of its arguments.

2. PURPOSE OF THE PAPER

Previously mentioned results concerning p(n, m; k) do not actually lead to asymptotic formulae in n, in the customary sense. If one replaces the parameters occurring in [5] or [8] by their asymptotic values as functions of n, m and k, one obtains (see [8], Corollaries 1 and 2) results of the form

$$p(n, m; k) = \tilde{p}(n, m; k) \exp\{O(n^{1/(k+1)} \log^c n)\},$$

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where $c = -2$ or $-2 + 1/(k+1)$ and $\tilde{p}$ is built up from elementary functions. Hence the relation, $\lim p(n, m;k)/\tilde{p}(n, m; k) = 1$ does not hold, in general. Here, and in what follows, it is assumed that $n \to \infty$ and, subject to specified restrictions, that also $m \to \infty$, unless the contrary is stated. The O- and o-notations are used with their customary meaning; square brackets stand for the greatest-integer function, and round brackets for the greatest common divisor. p and (sometimes) q stand for primes, and the logarithm is denoted by log. It is the purpose of this paper

(a) to express $p(n, m; k)$ as a sum of decreasing terms, starting with $\tilde{p}(n, m; k)$, already known, and, hence, to reduce the order of the error term;

(b) to establish for $p(n, m; k)$ formulae, depending on n, m and k, that are asymptotic in the customary sense.

3. ERROR TERMS—THREE POINTS OF VIEW

The formulae that will be established contain error terms, expressed in the O- or o-notation. Their meaning may be defined precisely, in different ways, as follows:

(i) One may consider the given values of m and n as belonging to a sequence of ordered pairs, all with the same m, while $n \to \infty$.

(ii) It may be convenient to consider (n, m) as an element of a sequence of ordered pairs (n_i, m_i) with $m_i = n_i^{\varepsilon_i}$, $m_i, n_i \to \infty$, $\varepsilon_i \to 0$.

(iii) Given m and n such that $m^{k+1} < n \log m$, we define $\lambda < 1$ by

$$m^{k+1} = n^{\lambda} \log m. \tag{1}$$

If $m^{k+1} > n \log n$, define $\mu > 1$ by

$$m^{k+1} = n \log^{\mu} n \tag{2}$$

(since m, n and k are integers, $m^{k+1} = n \log m$ cannot occur). In case

$$n \log m \le m^{k+1} < n \log n,$$

it is most convenient to consider

$$m^{k+1} = 2C\, n \log m \qquad (1 \le 2C < k+1). \tag{1'}$$

Then one may consider the given values m and n as belonging to a sequence of ordered pairs (m, n) with $m, n \to \infty$, while satisfying (1), (1') or (2), accordingly. The constants implied by the O- or o-terms may depend on k, λ, μ or C. For the same pair of values (m, n) $(m^{k+1} < n)$, one may take any of these three points of view. While the precision of the formulae decreases from (i) to (iii), the first two are "practical" only if m^{k+1} is much smaller than n; except for Sections 6 and 7, it will be assumed that (1), (1') or (2) holds.

4. PRINCIPAL RESULTS

If the trivial condition $m \leq n^{1/k}$ is replaced by (1), with $\lambda \leq (k+1)/(k+2)$, an asymptotic formula of the form

$$(3) \qquad p(n, m; k) = \tilde{p}_1(n, m; k)(1 + o(1)), \text{ with } \tilde{p}_1(n, m; k) = \frac{n^{M-1} \exp\{-k\,\theta(m)\}}{(M-1)!}$$

holds, where $\theta(m) = \Sigma \log p$ as usual, and $M = \pi(m) = \Sigma\, 1$. Here and throughout the paper, summations without indications of limits are understood over all primes $p \leq m$, unless the contrary is stated.

An asymptotic formula analogous to (3), giving the number of partitions of n into parts (not necessarily prime powers) at most equal to m as $\sim \dfrac{n^{m-1}}{m!(m-1)!}$ has been known for a long time (see for example Szekeres [10]). Relation (3) can easily be proved by elementary methods similar to those in [1], in case m is kept constant as $n \to \infty$. This approach succeeds also if $m, n \to \infty$, but $m \leq \log^h n$, for some $h < \infty$; this corresponds to (ii) in Section 3. If $k = 1$, the elementary method permits the proof of (3) whenever it is valid. Although the result is known for constant m (see [9, Vol. 1, Part 1, Problem 27, p. 4]; see also [7]), a short proof of it is given in Section 6, which owes much to [10]. The elementary proof of (3) for $m \to \infty$ is in Section 7. More precise results are obtained by nonelementary methods, by means of a Fundamental Lemma (denoted hereafter by FL), obtained by the saddle point method of integration (Section 4); this has already been used similarly in [3]. The main result is contained in

THEOREM 1.

(i) *If* m *is constant while* $n \to \infty$, *then*

$$(4) \qquad p(n, m; k) = \tilde{p}_1(n, m; k)(1 + \psi(n)),$$

where $\tilde{p}_1(n, m; k)$ *is given by* (3), *and where* $\psi(n) = O(n^{-1})$ *can be determined explicitly.*

(ii) *If* (1) *holds, then*

$$(4') \qquad p(n, m; k) = p_1(n, m; k)(1 + v(n, m; k) + E_1),$$

where $p_1(n, m; k)$ *is the function of the transcendental parameters* α, A_1, A_2 *(depending on* n, m, k*) defined by* (12), *and* $v(n, m; k) = v_1(m; k)$ *is defined by* (25); $E_1 = O(n^{-\gamma} \log^{-1/2} n)$, *where* γ *is finite but may be selected arbitrarily large.*

(iii) *If* (1) *holds with* $\lambda \leq (2k+2)/(2k+3)$, *then*

$$(4'') \qquad p(n, m; k) = (2\pi M)^{-1/2} \alpha^{1-M} \exp\{M - k\,\theta(m)\}\,(1 + v_1(m; k) + E_2),$$

where

$$\alpha = \frac{M}{n}\left(1 - \frac{m^{k+1}}{2(k+1)\, n \log m}\{1 + O(\log^{-1} m)\}\right),$$

$v_1(m; k)$ *is defined by* (25), *and* $E_2 = O\left(\dfrac{n^{-2+\lambda(2k+3)/(k+1)}}{(\log n)^{k/(k+1)}}\right)$.

(iv) *If* (1) *holds with* $\lambda \leq (k+1)/(k+2)$, *then*

$$p(n, m; k) = \tilde{p}_1(n, m; k)\left(1 + \frac{m^{k+2}}{2(k+1)\, n \log^2 m} + E_3\right), \tag{4'''}$$

where $\tilde{p}_1(n, m; k)$ *is given by* (3) *and* $E_3 = O\left(\dfrac{n^{-1+\lambda(k+2)/(k+1)}}{(\log n)^{(2k+1)/(k+1)}}\right)$.

(v) *If* (2) *holds, then* (4') *is valid with* $v = v_2(n; k)$ *defined by* (27) *and* $E_1 = O(n^{-\gamma} \log^{-1/2} n)$, *as under* (ii).

(vi) *The transcendental expression* (27) *for* $v_2(n; k)$ *may be approximated by*

$$v_2(n; k) = \frac{k}{8(k+1)} \left\{ \frac{\left(\Gamma\left(4 + \frac{1}{k}\right)\right)^{k+1} n^{-1} \log^k n}{\left(\Gamma\left(2 + \frac{1}{k}\right)\right)^{2k+1} \left(\zeta\left(1 + \frac{1}{k}\right)\right)^k} \right\}^{\frac{1}{k+1}}$$

$$- \frac{15}{2(6!)^2} \frac{k}{k+1} \left\{ \frac{\left(\Gamma\left(6 + \frac{1}{k}\right)\right)^{2k+2} n^{-1} (\log n)^{4-3k}}{\left(\Gamma\left(2 + \frac{1}{k}\right)\right)^{3k+2} \left(\zeta\left(1 + \frac{1}{k}\right)\right)^k} \right\}^{\frac{1}{k+1}} + O\left(\frac{\log \log n}{(n \log n)^{1/(k+1)}}\right).$$

The Fundamental Lemma (FL) is stated and proved in Section 5. After the elementary proofs of (3) in Sections 6 and 7 follows the proof of Theorem 1 by use of the FL; this occupies Sections 8-14. In Section 8, representations are found for $F(x)$, the generating function of the partitions, and for some of its derivatives. The position of the saddle point is examined in Section 9. Then follows the verification that $F(x)$ satisfies the hypotheses of the FL, the determination of $v(n, m; k)$, and the proof of (4') (Sections 10 to 12). The proof of Theorem 1 is finished in Sections 13 and 14, and the results for the most interesting particular case $k = 1$ are summarized in Section 15.

5. THE FUNDAMENTAL LEMMA

Except for Sections 6 and 7, where the approach of [1] is used, the method follows closely that of [3]; the integrals are evaluated by the saddle point method as in [3], [5] and [8]. The principal tool is the following lemma, which is essentially Theorem 12 in Hayman's paper [6]. The notation of [3] has been changed somewhat in order to facilitate a comparison with [8]. Results from [8] will be quoted freely as needed.

FUNDAMENTAL LEMMA. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *be analytic inside the unit circle and real on the interval* $0 \leq x < 1$. *Define the functions*

$$a(r) = r\frac{d(\log f(r))}{dr} \quad \text{and} \quad A_2(r) = r\frac{da(r)}{dr}$$

of $r = |x|$, *and assume that* $A_2(r) \to \infty$ *as* $r \to 1$. *Denote by* $\rho = \rho_n$ *the root of*

$$a(\rho) = n, \tag{5}$$

which approaches 1 *as* $n \to \infty$ (for the existence and even uniqueness of this root under slightly more general conditions, see [6, p. 72]). *Assume that for*

$$0 \leq r_0 < r < 1$$

there exist functions $\delta(r)$ *and* $u(r)$ *with the following properties:* As $n \to \infty$, *for some* $\omega \geq 0$ *and real* τ ($\tau < -1/2$ *if* $\omega = 0$),

(a) $\delta^2(\rho)\, A_2(\rho) \geq 2\omega \log n - \left(\tau + \frac{1}{2}\right) \log\log n$;

(b) $$\int_{-\delta(\rho)}^{\delta(\rho)} \left(f(\rho e^{i\theta}) - f(\rho) \exp\left\{ i\theta\, a(\rho) - \frac{1}{2}\theta^2 A_2(\rho) \right\} \right) e^{-in\theta}\, d\theta$$

$$= \left(\frac{2\pi}{A_2(\rho)} \right)^{1/2} f(\rho)\,(u(\rho) + O(n^{-\omega} \log^{\tau} n));$$

(c) $$\int_{|\theta| \geq \delta(\rho)} |f(\rho e^{i\theta})|\, d\theta = \frac{f(\rho)}{(A_2(\rho))^{1/2}}\, O(n^{-\omega} \log^{\tau} n).$$

Then

(6) $$a_n = \frac{f(\rho)}{\rho^n (2\pi A_2(\rho))^{1/2}} \{1 + u(\rho) + O(n^{-\omega} \log^{\tau} n)\}.$$

Remark. The possibility $u(\rho) = o(n^{-\omega} \log^{\tau} n)$ is not ruled out.

Proof of the FL. By Cauchy's theorem, integration around the circle of radius $r < 1$ gives

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(re^{i\theta})}{r^{n+1} e^{i(n+1)\theta}}\, rie^{i\theta}\, d\theta;$$

consequently,

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})\, e^{-in\theta}\, d\theta = \frac{1}{2\pi} \left[\int_{|\theta|>\delta} f(re^{i\theta})\, e^{-in\theta}\, d\theta \right.$$

$$+ \int_{-\delta}^{\delta} \left\{ f(re^{i\theta}) - f(r) \exp\left\{ i\theta\, a(r) - \frac{1}{2}\theta^2 A_2(r) \right\} \right\} e^{-in\theta}\, d\theta$$

$$\left. + f(r) \int_{-\delta}^{\delta} \exp\left\{ i\theta\,(a(r) - n) - \frac{1}{2}\theta^2 A_2(r) \right\} d\theta \right].$$

For $r = \rho$, it follows from (5) that the last integral becomes

$$\int_{-\delta}^{\delta} e^{-\theta^2 A_2/2}\, d\theta = \left(\frac{2}{A_2(\rho)} \right)^{1/2} \int_{-D}^{D} e^{-x^2}\, dx = \left(\frac{2}{A_2(\rho)} \right)^{1/2} (\pi^{1/2} - \varepsilon),$$

with $D = \delta(\rho)\,(A_2(\rho)/2)^{1/2}$ and $\varepsilon = \int_D^{\infty} e^{-x^2}\, dx < e^{-D^2}/2D$. The conclusion now follows from the assumptions made, and from the observation that (a) implies $0 < \varepsilon = O(n^{-\gamma} \log^{\tau} n)$.

6. THE ELEMENTARY APPROACH

The generating function of the $p(n, m; k)$ is

$$F(x) = \prod_{p\leq m} (1 - x^{p^k})^{-1} = \sum_{n=0}^{\infty} p(n, m; k)x^n. \tag{7}$$

Hence,

$$F(x) = (1 - x)^{-M} \prod_{p\leq m} \left(\sum_{\nu=0}^{p^k-1} x^{\nu}\right)^{-1}.$$

If p, q are distinct primes, then p^k, q^k are coprime; hence all zeros of $F(x)^{-1}$ are simple, except $x = 1$, and one has the decomposition into partial fractions

$$F(x) = \sum_{r=1}^{M} A_r (1 - x)^{-r} + \sum \sum_{1\leq \ell \leq p^k-1} a_{\ell}^{(p)} (1 - xe^{-2\pi i\ell/p^k}). \tag{8}$$

From $(1 - x)^{-r} = \sum_{n=0}^{\infty} \binom{n+1}{r-1} x^n$ it follows that in (8) the coefficient of x^n is

$$p(n, m; k) = \sum_{r=1}^{M} \binom{n+1}{r-1} A_r + \phi(n), \quad \text{with} \quad \phi(n) = \sum \sum_{1\leq \ell \leq p^k-1} a_{\ell}^{(p)} \exp\{2\pi in\ell/p^k\}.$$

In order to determine A_M, multiply (8) by $F(x)^{-1} = (1 - x)^M \prod_{p\leq m} \left(\sum_{\nu=0}^{p^k-1} x^{\nu}\right)$, obtaining

$$\begin{aligned} 1 = {} & \prod_{p\leq m} \left(\sum_{\nu=0}^{p^k-1} x^{\nu}\right) \sum_{r=1}^{M} A_r (1 - x)^{M-r} \\ & + (1 - x)^M \prod_{p\leq m} \left(\sum_{\nu=0}^{p^k-1} x^{\nu}\right) \sum \sum_{1\leq \ell \leq p^k-1} \frac{a_{\ell}^{(p)}}{1 - xe^{-2\pi i\ell/p^k}} \end{aligned} \tag{9}$$

Setting $x = 1$ in (9), one gets $A_M = e^{-k\theta(m)}$; hence,

$$p(n, m; k) = n^{M-1} e^{-k\theta(m)} \cdot \frac{1 + g(n)}{(M - 1)!} + \phi(n). \tag{10}$$

Here

$$g(n) = (\tilde{p}_1(n, m; k))^{-1} \sum_{r=1}^{M} \binom{n+1}{r-1} A_r - 1 = \sum_{s=1}^{M-1} \frac{\alpha_s}{n^s} = O(n^{-1}),$$

and all coefficients α_s can be determined explicitly. The function $\phi(n)$ is periodic, of period $e^{k\theta(m)}$.

If m is constant as $n \to \infty$, then $\phi(n)$ stays bounded and can be determined explicitly, for example, by setting $n = 1, 2, \cdots, e^{k\theta(m)}$ in (10) and using the periodicity of $\phi(n)$ for $n > e^{k\theta(m)}$. From (10) now follows (3) with o(1) replaced by the exactly known expression $\psi(n) = g(n) + \phi(n)(\tilde{p}_1)^{-1}$. This proves part (i) of Theorem 1.

7. THE ELEMENTARY APPROACH (CONTINUED)*

The same argument proves the validity of (3) whenever $\phi(n) = o(\tilde{p}_1)$. In the general case, much better results are obtained by use of the FL; also, the elementary argument becomes rather involved. Therefore, we shall consider in some detail only the case $k = 1$, with indications of the general situation. Let $\zeta = e^{2\pi i/p}$; then, setting $x = \zeta^\ell$ in (9), one obtains

$$1 = a_\ell^{(p)} (1 - \zeta^\ell)^M \prod_{\substack{q \neq p \\ q \leq m}} \left(\sum_{r=0}^{q-1} \zeta^{r\ell} \right) \prod_{\substack{1 \leq \nu \leq p-1 \\ \nu \neq \ell}} (1 - \zeta^\ell \zeta^{-\nu}) \qquad (1 \leq \ell \leq p - 1,\ q \text{ primes}).$$

Since $\sum_{r=0}^{p-1} \zeta^{r\ell} = 0$, $\sum_{r=0}^{q-1} \zeta^{r\ell} = \sum_{r=0}^{s-1} \zeta^{r\ell}$ with $s \equiv q \pmod p$. It is known that if $M_s = \sum_{\substack{q \leq m \\ q \equiv s \pmod p}} 1$, then $M_s = (M/(p-1))(1 + O(\log^{-v} m))$ for every $v < \infty$. Hence,

$$(1 - \zeta^\ell)^M \prod_{\substack{q \neq p \\ q \leq m}} \left(\sum_{r=0}^{q-1} \zeta^{r\ell} \right) = \prod_{s=1}^{p-1} (1 - \zeta^{\ell s})^{M_s} = \prod_{s=1}^{p-1} (1 - \zeta^{\ell s})^{M(1+\varepsilon)/(p-1)}.$$

Also, $|1 - \zeta^{\ell s}| = 2 \sin(\pi \ell s/p)$; hence, with the notation $M' = M(1 + \varepsilon)$,

$$1 = |a_\ell^{(p)}| \prod_{s=1}^{p-1} \left(2 \sin \frac{\pi \ell s}{p} \right)^{M'/(p-1)} \prod_{\nu=1}^{p-1} \left(2 \sin \frac{\pi \nu}{p} \right) \cdot \left(2 \sin \frac{\pi \ell}{p} \right)^{-1},$$

or

$$|a_\ell^{(p)}| = 2 \sin \frac{\pi \ell}{p} \prod_{s=1}^{(p-1)/2} \left(2 \sin \frac{\pi s}{p} \right)^N,$$

where $N = -2(M' + p - 1)/(p - 1)$. Consequently,

$$|a_\ell^{(p)}| \leq 2 \prod_{s=1}^{(p-1)/2} \left(\frac{4}{\pi} \frac{\pi s}{p} \right)^N = 2 \left(\left(\frac{p}{4} \right) \left(\left(\frac{p-1}{2} \right)! \right)^{-2/(p-1)} \right)^{M'+p-1}.$$

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* This section owes its existence to a suggestion of Prof. N. Fine.

Stirlings' formula shows that this is not greater than

$$2\left(\frac{e}{2p^{1/(p-1)}(1-1/p)^{p/(p-1)}}\right)^{M'+p-1}.$$

Except for $p = 2$, this in turn is less than $2e^{M'+p-1}p^{-1-M'/(p-1)}$; consequently, for $p > 2$,

$$\left|\sum_{1\le\ell\le p-1} a_\ell^{(p)}\zeta^{n\ell}\right| \le 2e^{M'+p-1}p^{-M'/(p-1)},$$

while $a_1^{(2)}$ is a constant. Hence,

$$|\phi(n)| = \left|\sum\sum_{1\le\ell\le p-1} a_\ell^{(p)}\zeta^{n\ell}\right| = O\left(\sum e^{M'+p-1}p^{-M'/(p-1)}\right)$$

$$= O\left(e^{M'}\int_2^m e^{x-1}(x-1)^{-M'/(x-1)}dx\right) = O\left(e^{M'}\int_1^m e^y y^{-M'/y}dy\right) = O(e^{M'+m}).$$

This will be $o(\tilde{p}_1(n, m; k))$, provided that

$$e^{M'+m} = o\left(n^{M-1}e^{-m}\cdot e^{O(m/\log m)}\cdot e^M M^{-M+\frac{1}{2}}\right).$$

If the exponent is divided by $m/\log m$ throughout, the observation that the ratios $M(\log m)/m$ and $M'(\log m)/m$ both tend to one shows that the estimate holds if $e^{1+\log m} = o(n^{1-1/M}e^{-\log m}(m/\log m)^{-1+1/2M})$, or, a fortiori, for

$$m^3 = o(n\log^{3/2} m)^{1-1/M}. \tag{11}$$

For $k = 1$, (11) is essentially the condition $\lambda \le (k+1)/(k+2)$ of Theorem 1 (iv), needed for the validity of the asymptotic formula (3). In exactly the same way, one verifies that if $m^3 = O(n^{1-2m^{-1}\log m})$, then $\phi(n) = O(n^{-1}\cdot\tilde{p}_1)$. In that case, that is for every $\lambda < 2/3$, (4) holds with some (not explicitly known) $\psi(n) = O(n^{-1})$.

In case $k > 1$, it is necessary to use the fact that not only the primes q, but also their k-th powers are essentially evenly distributed among the $t = p^{k-1}(p-1)/d$ [here $d = (k, p^{k-1}(p-1))$] residue classes mod p^k, where they occur. Also, the previous rough estimate of $|\phi(n)|$ becomes insufficient and yields only the condition

$$\exp\{m^{k-1}\log m\} = o(n^{1-1/M}e^{1-k\log m}m^{(k-1)/M}(m/\log m)^{-1+1/2M}). \tag{11'}$$

Although (11) follows from (11') for $k = 1$, the latter condition is in the general case a much more stringent one and permits to infer the validity of (3) only if $m = o(n^\varepsilon)$ for every $\varepsilon > o$. In order to overcome this difficulty, one has to use a rather sophisticated method for the estimation of $\phi(n)$, as can be seen, for example, from Szekeres' handling of a similar problem [10]. Instead of this it seems preferable to make straightforward use of the FL.

8. AN AUXILIARY THEOREM

Let $A_2(\rho) = A_2$, $\log \rho = -\alpha$ and $\log F(\rho) = A_1$; then the following theorem holds:

THEOREM 2. *The function* F(x) *defined by* (7) *satisfies the hypotheses of the FL, so that, for appropriate* $u(\rho)$, ω *and* τ,

$$p(n, m; k) = p_1(n, m; k)\,(1 + u(\rho) + O(n^{-\omega} \log^{\tau} n)), \tag{12}$$

where $p_1(n, m; k) = (2\pi A_2)^{-1/2} \exp\{n\alpha + A_1\}$, $u(\rho) = o(1)$.

It is sufficient to verify that F(x) satisfies the hypotheses of the FL, since (12) then follows directly from (6). Also, Theorem 1 is an immediate consequence of Theorem 2. In $p_1(n, m; k)$, one recognizes the asymptotic expression of Haselgrove and Temperley [5] and of Mitsui [8].

The proof of Theorem 2 will now begin; it will run to the end of Section 12.

From (7), with $-\beta = \log x$, $\Re\beta > o$, one obtains $\log F(x) = \Sigma \Sigma_{\nu=1}^{\infty} \nu^{-1} \exp(-\nu\beta p^k)$. By Mellin's theorem, the exponential equals

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} (\nu\beta p^k)^{-s}\,\Gamma(s)\,ds \qquad (s = \sigma + it,\ \sigma > o),$$

so that $\log F(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \beta^{-s}\Gamma(x)\,\zeta(s+1)\,(\Sigma p^{-ks})ds$. The interchange in the order of summation and integration is justified by the uniformity of convergence. Changing s into -s, one obtains

$$\log F(x) = \frac{1}{2\pi i}\int_{-\sigma-i\infty}^{-\sigma+i\infty} \beta^s\,\Gamma(-s)\,\zeta(1-s)\sum p^{ks}\,ds = \frac{i}{2\pi}\int_{-\sigma-i\infty}^{-\sigma+i\infty} \beta^s \frac{\Gamma(1-s)\,\zeta(1-s)}{s}\sum p^{ks}\,ds$$

$$= \frac{i}{2}\int_{-\sigma-i\infty}^{-\sigma+i\infty} \left(\frac{\beta}{2\pi}\right)^s \frac{\zeta(s)}{s\sin\frac{\pi s}{2}}\sum p^{ks}\,ds,$$

the last two equalities being justified by the functional equations of the Γ- and the ζ-function, respectively. The integrand is meromorphic, with poles at s = 0, 1, 2j (j = 1, 2, ⋯). The corresponding residues are found to be

$$R_0 = \left\{M \log\frac{\beta}{2\pi} + M\zeta'(0)/\zeta(0) + k\,\theta(m)\right\}\frac{2}{\pi}\,\zeta(0) = -\{M\log\beta + k\,\theta(m)\}/\pi,$$

$$R_1 = \frac{\beta}{2\pi}\sum p^k \quad\text{and}\quad R_{2j} = \frac{(-1)^j}{\pi}\left(\frac{\beta}{2\pi}\right)^{2j}\frac{\zeta(2j)}{j}\sum p^{2kj}.$$

Consequently,

$$\begin{aligned}\log F(x) = &-(M\log\beta + k\,\theta(m)) + \tfrac{1}{2}\beta\sum p^k + \sum_{j=1}^{L}(-1)^j\left(\frac{\beta}{2\pi}\right)^{2j}\frac{\zeta(2j)}{j}\sum p^{2kj}\\ &+\frac{i}{2}\int_{h-i\infty}^{h+i\infty}\left(\frac{\beta}{2\pi}\right)^s\frac{\zeta(s)}{s\sin\frac{\pi s}{2}}\left(\sum p^{ks}\right)ds,\end{aligned} \tag{13}$$

with $2L + \varepsilon \leq h \leq 2L + 2 - \varepsilon$ $(0 < \varepsilon < 1)$ for any positive integer L and $1 + \varepsilon \leq h \leq 2 - \varepsilon$ for $L = 0$.

Consider F(x) as a function of β, and set $\log F(x) = f(\beta)$. Differentiation gives

$$(14)\quad \begin{aligned} a(x) = \frac{d \log F(x)}{d \log x} = -f'(\beta) = \frac{M}{\beta} - \frac{1}{2}\sum p^k - \frac{1}{\pi}\sum_{j=1}^{L} (-1)^j \left(\frac{\beta}{2\pi}\right)^{2j-1} \zeta(2j) \sum p^{2kj} \\ + \frac{1}{4\pi i}\int_{h-i\infty}^{h+i\infty} \left(\frac{\beta}{2\pi}\right)^{s-1} \frac{\zeta(s)}{\sin\frac{\pi s}{2}} \left(\sum p^{ks}\right) ds, \end{aligned}$$

$$(15)\quad \begin{aligned} \frac{d\,a(x)}{d \log x} = A_2(x) = f''(\beta) = \frac{M}{\beta^2} + \frac{1}{2\pi^2}\sum_{j=1}^{L} (-1)^j \left(\frac{\beta}{2\pi}\right)^{2j-2} (2j-1)\,\zeta(2j) \sum p^{2kj} \\ - \frac{1}{8\pi^2 i}\int_{h-i\infty}^{h+i\infty} \left(\frac{\beta}{2\pi}\right)^{s-2} \frac{(s-1)\,\zeta(s)}{\sin\frac{\pi s}{2}} \left(\sum p^{ks}\right) ds. \end{aligned}$$

9. THE SADDLE POINT

The representation (14), in conjunction with (7), permits us to determine the root $\alpha = -\log \rho$ of (5), the "saddle point" used in the FL. It turns out that, for fixed n, α increases with m, until it reaches the value

$$\alpha = \alpha_o = \left(\frac{\Gamma\left(2 + \frac{1}{k}\right)\zeta\left(1 + \frac{1}{k}\right)}{n \log n}\right)^{k/(k+1)} \left(1 - \frac{k \log\log n}{(k+1)\log n} + O(\log^{-1} n)\right).$$

Afterwards, for $m^{k+1} > n (\log n)^{1+\varepsilon}$, α becomes independent of m.

THEOREM 3. *Let* $m^{k+1} = n^\lambda \log^\nu m$, *with* $\lambda \leq 1$ *and with* $\nu = 1$ *if* $\lambda < 1$. *Set* $\mu = \min(1, \nu)$. *Then*

$$(16)\qquad \alpha = C_{\lambda,\nu}\left(n^{\lambda-k-1} (\log n)^{\mu-k-1}\right)^{1/(k+1)} (1 + o(1)),$$

with

$$C_{\lambda,\nu} = \left(\frac{\lambda}{k+1}\right)^{(\mu-k-1)/(k+1)} \qquad \text{if } \lambda\nu < 1,$$

$$C_{1,1} = (k+1)^{(2k+1)/(k+1)} (k + 3/2)^{-1},$$

$$C_{1,\nu} = F_k = \{\Gamma(2 + 1/k)\zeta(1 + 1/k)\}^{k/(k+1)} \qquad \text{if } \nu > 1.$$

In the proof of Theorem 3 and frequently thereafter, we shall need

LEMMA 1. $\sum p^r = \dfrac{m^{r+1}}{(r+1)\log m}\left(1 + O\left(\dfrac{1}{\log m}\right)\right)$.

Proof of Lemma 1.

$$\sum p^r = \sum r \int_0^p x^{r-1}\,dx = r \sum_{1 \le \nu \le m} \{\pi(m) - \pi(\nu - 1)\} \int_{\nu-1}^{\nu} x^{r-1}\,dx$$

$$= r\,\pi(m) \int_0^m x^{r-1}\,dx - r \int_2^m \pi(x)\,x^{r-1}\,dx = \pi(m)\,m^r - r \int_2^m \left(\frac{x^r}{\log x} + O\left(\frac{x^r}{\log^2 x}\right)\right)dx$$

$$= \pi(m)\,m^r - \frac{r}{r+1}\,m^{r+1}\log^{-1} m + O\left(\frac{m^{r+1}}{\log^2 m}\right) = \frac{m^{r+1}}{(r+1)\log m}\left(1 + O\left(\frac{1}{\log m}\right)\right).$$

Proof of Theorem 3. With $L = 0$ and $h = 1 + \varepsilon$ in (14), the Riemann-Lebesgue theorem and Lemma 1 show that for $m^{k+1} = n^\lambda \log^\nu m$ and $\alpha = O((n \log n)^{-k/(k+1)})$, the integral is

$$o(n^{\lambda + (\lambda-1)k\varepsilon/(k+1)} (\log n)^{-1+(\nu-1)k\varepsilon/(k+1)}) = o(n).$$

In particular, if (1) holds, then by Lemma 1,

$$\sum p^k = \frac{m^{k+1}}{(k+1)\log m}\left(1 + O\left(\frac{1}{\log m}\right)\right) = O(n^\lambda).$$

Also, with $L = 1$ and $h = 2 + \varepsilon$ in (14), the integral along $h = 1 + \varepsilon$ is seen to be actually $O(n^{\lambda_2})$, where $\lambda_2 = \{(2k+1)\lambda - k\}/(k+1) < \lambda$. (It does not seem possible to obtain this result directly, with $L = 0$, since the o-estimate does not hold uniformly in ε.) Equation (5) becomes

$$\frac{M}{\beta} - \frac{m^{k+1}}{2(k+1)\log m}(1 + O(\log^{-1} m)) = n(1 + O(n^{\lambda_2 - 1})),$$

whence

(16')
$$\alpha = \frac{M}{n}\left(1 - \frac{m^{k+1}}{2(k+1)\,n \log m}(1 + O(\log^{-1} m))\right)$$

and (16) holds in this case.

If $m^{k+1} = n \log^\mu m$ ($\mu < 1$), then $\Sigma p^k = O(n \log^{\mu-1} n) = o(n)$, and by (5) and (14),

(16'')
$$\alpha = \frac{M}{n}(1 + o(1)),$$

and (16) holds again.

If (1') holds, the integral along $h = 1 + \varepsilon$ is still $o(n)$, but now

$$\frac{1}{2}\sum p^k = \frac{m^{k+1}}{2(k+1)\log m}(1 + O(\log^{-1} m)) = \frac{Cn(1 + o(1))}{k+1},$$

and from (5) follows

(16''')
$$\alpha = \frac{k+1}{C+k+1}\,\frac{M}{n}(1 + o(1)) = \frac{F}{(n \log n)^{k/(k+1)}}(1 + o(1)), \quad \text{where}$$

$$F = \frac{(k+1)^2}{C+k+1}\left(\frac{2C}{k+1}\right)^{1/(k+1)},$$

and (16) follows from (16''') for $C = 1/2$.

Remark. Since $C \neq 0, \infty$ in (1'), log m in the denominator may be replaced by log n, by renaming the constant C.

The proof of (16) for $\nu > 1$ is less immediate. For $\nu \geq 2$, Mitsui has shown (see [8]) that

$$(16'''')\qquad \alpha = \alpha_o = \frac{F_k}{(n \log n)^{k/(k+1)}}\left(1 - \frac{k \log\log n}{(k+1)\log n} + O(\log^{-1} n)\right), \quad \text{where}$$

$$F_k = \left(\Gamma\left(2+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\right)^{k/(k+1)}.$$

A slight modification of his proof shows the validity of (16'''') for any $\nu > 1$; this will finish the proof of Theorem 3. For that, one first has to replace Lemma 1 of [8] by

LEMMA 2. *If* $m^{k+1} = n \log^{\nu} n$ $(1 < \nu \leq 2)$, *then* $\alpha = \dfrac{c_{k,\nu}}{(n \log n)^{k/(k+1)}}$, *where* $\frac{1}{2}(1+\varepsilon_1) < c_{k,\nu} < \{6(k+1)\}^{k/(k+1)}(1+\varepsilon_2)$, $\varepsilon_1, \varepsilon_2 \to 0$ *as* $n \to \infty$.

Proof of Lemma 2. For some $b > o$, set

$$\beta = \frac{b}{n}\min\{(n(\log n)^{\nu-k-1})^{1/(k+1)}, m \log^{-1} m\}.$$

By the assumption in the lemma, $m \log^{-1} m \sim (k+1)(n(\log n)^{\nu-k-1})^{1/(k+1)}$; therefore,

$$\beta = b\,(n^{-k}(\log n)^{\nu-1-k})^{1/(k+1)} = O(n^{-k/(k+1)}).$$

Hence, $m\beta^{1/k} = (b \log^{\nu-1} n)^{1/k}$. We shall be interested in values of b (not necessarily constant), such that $b(\log n)^{\nu-1} > 1$. Then $\min(1, m\beta^{1/k}) = 1$ and, with the value of β as here defined, (10) of [8] reads

$$\frac{1+\varepsilon_2}{2\beta^{1+1/k}\log m} < a(x) = \sum \frac{p}{e^{\beta p^k} - 1} < \frac{6(1+\varepsilon_1)}{\beta^{1+1/k}\log m};$$

here the equality follows by differentiation of (7). Replacement of β by its value gives

$$a(x) < 6(1+\varepsilon_1)\log^{-1} m \cdot b^{-(k+1)/k} n(\log n)^{(k-\nu+1)/k},$$

and this is less than n, provided that

$$b > b_1 = \{6(1+\varepsilon_1)(k+1)\}^{k/(k+1)} (\log n)^{-(\nu-1)/(k+1)}.$$

For such values of b, one verifies that the assumption made,

$$b \log^{\nu-1} n = (6(k+1))^{k/(k+1)} (\log n)^{(\nu-1)k/(k+1)} > 1,$$

actually holds, provided that $\nu \geq 1$. It follows that, in order to have $a(x) = n$, the root α is obtained from the expression of β with some $b < b_1$, that is

$$\alpha \leq b_1 \{n^{-k}(\log n)^{\nu-1-k}\}^{1/(k+1)} = \{6(1+\varepsilon_1)(k+1)\}^{k/(k+1)}(n \log n)^{-k/(k+1)},$$

for every $\varepsilon_1 > 0$; this proves one inequality of the lemma. The other inequality is obtained in a similar (but simpler) manner. On the basis of Lemma 2 instead of Lemma 1 of [8], the proof of (16'''') proceeds for $1 < \nu \leq 2$ exactly as for $\nu \geq 2$ in [8], except for the difference that a certain error term now reads

$$O(\exp\{-c(\log n)^{(\nu-1)k/(k+1)}\}),$$

instead of $O(\exp\{-c(\log n)^{k/(k+1)}\})$; this does not alter the conclusion, provided (and it is here that the assumption is fully used) that $\nu > 1$. The proof of Theorem 3 is now complete.

COROLLARY 1. *If* $\lambda < 1$, *then the series in* (13), (14) *and* (15) *converge and the integrals tend to zero, as* L *and* $h \to \infty$.

Proof. By Lemma 1 and (16'), the ratio of consecutive terms in the series is at most $(\alpha m^k/2\pi)^2 = O(n^{\lambda-1} \log m) \to 0$. Also, by Lemma 1 and the Riemann-Lebesgue theorem, the integrals admit estimates of the form $o(n^{c-h(1-\lambda)} \log^{\xi} n)$, with c, ξ constants (different in each integral); hence, for $h \to \infty$, the integrals tend to 0.

COROLLARY 2. (i) *For every* $\varepsilon > 0$ *and* $\lambda < 1$,

$$A_2 = A_2(\rho) = \frac{M}{\alpha^2}(1 + o(n^{-(2-\varepsilon)(1-\lambda)}));$$

(ii)

$$A_2 = A_2(\rho) = G_{\lambda,\nu} n^{2-\lambda/(k+1)}(\log n)^{1-\mu/(k+1)}(1 + o(1)),$$

where $G_{\lambda,1} = C_{\lambda,1}^{-1}$ *if* $\lambda < 1$, $G_{1,\nu} = (1 + 1/k)C_{1,\nu}^{-1}$ *if* $\nu > 1$, $G_{1,1} = O(1)$ *if* (1') *holds*,

Proof. (i) is obtained on substituting (16) in the integral of (15), with $h = 2 - \varepsilon$, $\varepsilon > 0$. (ii) is obtained on substituting (16) in (i), when $\lambda < 1$; as a generalization of (46) in [8], by use of Lemma 2, when $\nu > 1$; finally, on substituting (16''') and m from (1') in the integral of (15), with $0 < h < 2$, when (1') holds.

10. PROOF OF THEOREM 2 (I)

Let $\delta = \delta(\rho)$ be defined by

(17) $$\delta^2(\rho) A_2(\rho) = 2\gamma \log n,$$

where $\gamma > 0$ is arbitrarily large, but finite. Condition (a) of the FL then holds for every τ if $\omega < \gamma$, and also for $\omega = \gamma$, if $\tau \geq -1/2$.

It will be easy to verify hypothesis (c); therefore, the main remaining difficulty is the determination of a function $u(\rho)$, for which (b) of the FL holds. For that, we need $\log F(\rho e^{i\theta})$. When $x = \rho e^{i\theta}$, $\log(1/x) = \alpha - i\theta$; hence, one obtains $\log F(\rho e^{i\theta}) = f(\beta)$, by setting $\beta = \alpha - i\theta$ in (13). Generally,

$$f(\alpha - i\theta) = \sum_{\nu=0}^{\infty} (-i\theta)^{\nu} f^{(\nu)}(\alpha)/\nu! = \sum_{\nu=0}^{\infty} (-i\theta/\alpha)^{\nu} f_{\nu},$$

where $f_{\nu} = \alpha^{\nu} f^{(\nu)}(\alpha)/\nu!$. In particular,

$$f_o = \log F(\rho) = A_1, \qquad f_1 = -a(\rho), \qquad f_2 = \frac{\alpha^2}{2} A_2(\rho);$$

hence,

(13')
$$\log F(\rho e^{i\theta}) = \log F(\rho) + i\theta\, a(\rho) - \frac{1}{2}\theta^2 A_2 + \sum_{\nu=3}^{\infty} (-i\theta/\alpha)^{\nu} f_{\nu}.$$

Denoting by $D(\rho, \theta)$ the integrand $\left\{F(\rho e^{i\theta}) - F(\rho)\exp(i\theta\, a(\rho) - \frac{1}{2}\theta^2 A_2)\right\} e^{-in\theta}$ in (b) of the FL, and remembering the definition of ρ as solution of (5), one obtains

$$D(\rho, \theta) = F(\rho)\exp\left(-\frac{1}{2}\theta^2 A_2\right)\left\{\exp\left(\sum_{\nu=3}^{\infty} (-i\theta/\alpha)^{\nu} f_{\nu}\right) - 1\right\}.$$

From here on, the procedure is essentially as follows: One obtains estimates for f_{ν}, uniformly in ν; this permits an estimate of $\Sigma_{\nu=N+1}^{\infty} |\theta/\alpha|^{\nu} f_{\nu}$ for $|\theta| \leq \delta$ and an arbitrary integer N. When it is shown that this sum is $o\{(\theta/\alpha)^N f_N\}$, the infinite series in $D(\rho, \theta)$ can be replaced by $\Sigma'^{N}_{\nu=3} (-i\theta/\alpha)^{\nu} f_{\nu}$, the dash indicating that f_N has to be replaced by $f'_N = f_N(1 + o(1))$, to compensate for the truncation. Next, one estimates $S = \Sigma'^{N}_{\nu=3}(-i\theta/\alpha)^{\nu} f_{\nu}$ and $|\Sigma_{\mu=Q+1}^{\infty} S^{\mu}/\mu!|$. If Q is selected in an appropriate way, the last sum can also be written as $(\theta/\alpha)^N f_N \cdot o(1)$, and $D(\rho, \theta)$ becomes

$$D(\rho, \theta) = F(\rho)\cdot\exp\left(-\frac{1}{2}\theta^2 A_2\right)\cdot \sum_{\mu=1}^{Q}{}' \frac{1}{\mu!}\left(\sum_{\nu=3}^{N}{}' \left(\frac{-i\theta}{\alpha}\right)^{\nu} f_{\nu}\right)^{\mu}$$

$$= F(\rho)\exp\left(-\frac{1}{2}\theta^2 A_2\right)\left(\sum_{j=2}^{NQ/2} (-1)^j \left(\frac{\theta}{\alpha}\right)^{2j} B_{2j} + f_1(\theta)\right).$$

Here $f_1(\theta)$ stands for an odd function and, by the multinomial theorem,

(18)
$$B_{2j} = \sum f_{\nu_1}^{\alpha} \cdots f_{\nu_r}^{\alpha_r} / \alpha_1! \cdots \alpha_r!,$$

the sum being extended over the integers ν and α satisfying the conditions

$$1 \leq \alpha_1 + \alpha_2 + \cdots + \alpha_r \leq g = [2j\ 3], \qquad \alpha_1 \nu_1 + \cdots + \alpha_r \nu_r = 2j,$$

$$3 \leq \nu_1 < \nu_2 < \cdots < \nu_r \leq N.$$

The integral of $f_1(\theta)$ over $(-\delta, +\delta)$ vanishes, and

$$\int_{-\delta}^{\delta} \theta^{2j}\exp\left(-\frac{1}{2}\theta^2 A_2\right)d\theta = 2A_2^{-(j+1/2)}\int_{o}^{\delta A_2^{-1/2}} u^{2j} e^{-u^2/2} du$$

$$= 2A_2^{-(j+1/2)}\left(\int_{o}^{\infty} u^{2j} e^{-u^2/2} du - \int_{\delta A_2^{-1/2}}^{\infty} \cdots\right).$$

The last integral is $O((\delta A_2^{1/2})^{2j-1} e^{-\delta^2 A_2/2}) = O(n^{-\gamma}(\log n)^{j-1/2})$, while the first, integrated by parts, equals $1\cdot 3\cdot 5 \cdots (2j-1)(\pi/2)^{1/2}$. Hence,

$$\int_{-\delta}^{\delta} D(\rho,\theta)\,d\theta = F(\rho)\left(\frac{2\pi}{A_2}\right)^{1/2} \sum_{j=2}^{NQ/2} (-1)^j \frac{B_{2j}}{\alpha^{2j}A_2^j}\frac{(2j)!}{2^j j!}\left(1 + O\left(n^{-\gamma}(\log n)^{j-\frac{1}{2}}\right)\right)$$

(19)
$$= F(\rho)\left(\frac{2\pi}{A_2}\right)^{1/2}(u(\rho) + O(n^{-\gamma-\gamma_1}\log^{\xi} n))$$

with

(20)
$$u(\rho) = \sum_{j=2}^{NQ/2} (-1)^j \frac{B_{2j}}{\alpha^{2j}A_2^j}\frac{(2j)!}{2^j j!}.$$

The values of γ_1 and ξ will depend on estimates of the B_{2j}. This will prove that hypothesis (b) of the FL is satisfied by $F(x)$.

11. ESTIMATES FOR f_ν

LEMMA 3. (i) *Let* $\bar{\nu} = [(\nu + 1)/2]$; *then, if* (1) *holds,* $f_\nu = (-1)^\nu \frac{M}{\nu} c_\nu$, *with*

(21)
$$c_\nu = 1 + (-1)^\nu \frac{\nu}{M}\sum_{j=\bar{\nu}}^{\infty} (-1)^j \binom{2j}{\nu}\left(\frac{\alpha}{2\pi}\right)^{2j}\frac{\zeta(2j)}{j}\sum p^{2kj}.$$

The estimate

(21')
$$|c_\nu - 1| = O(n^{-4(1-\lambda)})$$

holds uniformly in ν, *as* $n \to \infty$.

(ii) *If* (1') *holds, then* $f_\nu = (-1)^\nu \frac{M}{\nu} c_\nu$ *with* $c_\nu = O(1)$, *uniformly in* ν.

(iii) *If* (2) *holds, there exist constants* c *and* C, *independent of* ν, *such that, as* $n \to \infty$,

$$\frac{c}{\log n} < |\alpha^{1/k} f_\nu| < C.$$

(iv) *If* (2) *holds and* $n \to \infty$, *then, for any fixed* ν,

$$f_\nu = (-1)^\nu \frac{\Gamma\left(\nu + \frac{1}{k}\right)}{\nu!}\frac{\zeta\left(1 + \frac{1}{k}\right)}{\alpha^{1/k}\log(1/\alpha)}\left(1 - \frac{1}{\log(1/\alpha)}L_\nu + O(\log^{-2}(1/\alpha))\right),$$

with $L_\nu = J_\nu/\Gamma\left(\nu + \frac{1}{k}\right)\zeta\left(1 + \frac{1}{k}\right)$, $J_1 = \int_0^\infty \frac{u^{1/k}\log u}{e^u - 1}du$ *and*

(22)
$$J_\nu = -\left(\nu - 1 + \frac{1}{k}\right)J_{\nu-1} + (-1)^\nu \Gamma\left(\nu - 1 + \frac{1}{k}\right)\zeta\left(1 + \frac{1}{k}\right).$$

Remarks. 1. The case distinction is exhaustive and mutually exclusive. 2. The constant implied by the O-term in (iv) depends on ν.

Proof of Lemma 3. (i) By Corollary 1, if (1) holds, it is legitimate to set $L = \infty$ in (13), (14), (15). By part (i) of Corollary 2, $A_2 = \alpha^{-2}M(1 + o(1))$; hence, by (17), $\delta = \alpha(2\gamma \log n \cdot M^{-1})^{1/2}$, so that

(23) $$\frac{\delta}{\alpha} = \left(\frac{2\gamma \log n}{M}\right)^{1/2} = O(n^{-\lambda/2(k+1)} (\log n)^{(2k+1)/(2k+2)}).$$

For $\beta = \alpha - i\theta$ and $L = \infty$, (13) reads

$$\log F(\rho e^{i\theta}) = M\left(\log\frac{1}{\alpha} - \log\left(1 - \frac{i\theta}{\alpha}\right)\right) + \frac{1}{2}(\alpha - i\theta)\sum p^k - k\theta(m)$$

$$+ \sum_{j=1}^{\infty} (-1)^j \left(\frac{\alpha}{2\pi}\right)^{2j} j^{-1}\zeta(2j)\left(1 - \frac{i\theta}{\alpha}\right)^{2j} \sum p^{2kj}.$$

(There should be no danger of confusion between the arithmetic function $\theta(m) = \Sigma \log p$ and the angle θ.) With expansion of the logarithm and the binomials, (14) and (15) give

$$\log F(\rho e^{i\theta}) = M \log\frac{1}{\alpha} - k\,\theta(m) + \frac{\alpha}{2}\sum p^k$$

$$+ \sum_{j=1}^{\infty} (-1)^j \left(\frac{\alpha}{2\pi}\right)^{2j} j^{-1} \zeta(2j)\sum p^{2kj} + M\sum_{\nu=1}^{\infty}\left(\frac{i\theta}{\alpha}\right)^{\nu} \nu^{-1}$$

$$- \frac{i\theta}{2}\sum p^k + \sum_{j=1}^{\infty} (-1)^j \left(\frac{\alpha}{2\pi}\right)^{2j}\left(\sum_{\nu=1}^{2j}\binom{2j}{\nu}\left(\frac{-i\theta}{\alpha}\right)^{\nu}\right)\frac{\zeta(2j)}{j}\sum p^{2kj}$$

$$= \log F(\rho) + i\theta\alpha(\rho) - \frac{1}{2}\theta^2 A_2(\rho) + M\sum_{\nu=3}^{\infty}\left(\frac{i\theta}{\alpha}\right)^{\nu} \nu^{-1}$$

$$+ \sum_{j=1}^{\infty} (-1)^j \left(\frac{\alpha}{2\pi}\right)^{2j}\left(\sum_{\nu=3}^{2j}\binom{2j}{\nu}\left(\frac{-i\theta}{\alpha}\right)^{\nu}\right)\frac{\zeta(2j)}{j}\sum p^{2kj}.$$

Regrouping of terms yields

$$\log F(\rho e^{i\theta}) = \log F(\rho) + i\theta\, a(\rho) - \frac{1}{2}\theta^2 A_2 + \sum_{\nu=3}^{\infty}\left(\frac{i\theta}{\alpha}\right)^{\nu}\frac{M}{\nu} c_\nu,$$

with c_ν given by (21). By (15), (1) and Lemma 1, direct computation from (21) leads to $|c_\nu - 1| = O(n^{-2\overline{\nu}(1-\lambda)})$. Since $\nu \geq 3$, this implies (21'). This proves the first part of the lemma.

(ii) If in (13) one lets $L \to \infty$, with $h = L + 1$ and constant n, then, if (1') holds, on account of (16''') and the Riemann-Lebesgue theorem, the integral is $o(h^{-1}(n \log^{-k} n)^{1/(k+1)})$ and goes to zero as L and $h \to \infty$. Writing (13) for $L = L_1$, $h_1 = L_1 + 1$ and then for $L = L_2$, $h_2 = L_2 + 1$, one sees by subtraction that the sum from $j = L_1 + 1$ to L_2 is less in absolute value than the mean of the absolute values of the integrals along $h = h_1$ and $h = h_2$, respectively. Since these integrals go to zero, it follows by Cauchy's criterion that the series converges. Hence, the analysis from (i) is still valid. The result now follows on setting $\lambda = 1$ in (21'). For n, $m \to \infty$, one can no longer infer that $c_\nu \to 1$. In fact, this is no longer true. Indeed, one may compute, say, c_3, for $m^{k+1} = n \log m$, by using (21); one obtains $c_3 = 1 - 240(4k+1))^{-1}(1 + O(\log^{-1} m) \not\to 1$.

(iii) The series that one obtains by setting $L = \infty$ in (13) fails to converge if (2) holds. Therefore, one has to use the first representation of $F(x)$, in (7), $f(\alpha) = \log F(\rho) = -\sum \log(1 - e^{-\alpha p^k})$. Differentiation leads to

$$\frac{df}{d\alpha} = -\sum \frac{p^k}{e^{\alpha p^k} - 1} = -\sum p^k \left(\frac{1}{e^u - 1}\right)_{u=\alpha p^k}$$

and, by induction on ν,

$$\frac{d^\nu f}{d\alpha^\nu} = -\sum p^{\nu k} \left\{\frac{d^{\nu-1}}{du^{\nu-1}}\left(\frac{1}{e^u - 1}\right)\right\}_{u=\alpha p^k},$$

whence

$$f_\nu = \frac{\alpha^\nu}{\nu!} f^{(\nu)}(\alpha) = -\frac{1}{\nu!}\sum\left(u^\nu \frac{d^{\nu-1}}{du^{\nu-1}}\left(\frac{1}{e^u - 1}\right)\right)_{u=\alpha p^k}.$$

Since $\dfrac{d^{\nu-1}}{du^{\nu-1}}\left(\dfrac{1}{e^u - 1}\right) = (-1)^{\nu-1} \sum\limits_{\mu=1}^{\infty} \mu^{\nu-1} e^{-\mu u}$,

$$(-1)^\nu \nu!\, f_\nu = \sum_{p \le m}\left(\sum_{\mu=1}^{\infty} \frac{1}{\mu}(\mu u)^\nu e^{-\mu u}\right)_{u=\alpha p^k} < \sum_{n=2}^{m}\sum_{\mu=1}^{\infty} \frac{1}{\mu}(\mu\alpha n^k)^\nu e^{-\mu\alpha n^k}$$

$$= O\left(\sum_{\mu=1}^{\infty} \frac{1}{\mu}\int_1^\infty (\mu\alpha x^k)^\nu e^{-\mu\alpha x^k} dx\right) = O\left(\sum_{\mu=1}^{\infty} \frac{1}{\mu^{1+1/k} k\alpha^{1/k}} \int_{\mu\alpha}^\infty y^{\nu+\frac{1}{k}-1} e^{-y} dy\right)$$

$$= O\left(\frac{\Gamma\left(\nu + \frac{1}{k}\right)\zeta\left(1 + \frac{1}{k}\right)}{k\alpha^{1/k}}\right)$$

and $|f_\nu| < \dfrac{\Gamma\left(\nu + \frac{1}{k}\right)}{\nu!} \dfrac{\zeta\left(1 + \frac{1}{k}\right)}{k\alpha^{1/k}} O(1)$, and this proves the second inequality in (iii).

On the other hand,

$$\sum_{p=2}^{m}\sum_{\mu=1}^{\infty} \frac{1}{\mu}(\mu\alpha p^k)^\nu e^{-\mu\alpha p^k} \ge \frac{c}{\log m}\sum_{\mu=1}^{\infty}\frac{1}{\mu}\int_1^\infty (\mu\alpha x^k)^\nu e^{-\mu\alpha x^k} dx > \frac{c}{\log n} \frac{\nu!\,\zeta\left(1 + \frac{1}{k}\right)}{k\alpha^{1/k}};$$

this completes the proof of (iii).

(iv) It is known (see (7) in [8]) that for $r \ge s \ge 1$,

$$\sum \frac{p^{rk}}{(e^{\beta p^k} - 1)^s} = \int_2^m \frac{x^{rk} dx}{(e^{\beta p^k} - 1)^s \log x} + O(e^{-c \log^{1/2} m})\beta^{-r-\frac{1}{k}} \min(1, m\beta^{1/k}).$$

Using this result, decomposition into partial fractions, and Theorem 3, one obtains successively,

$$\frac{d^\nu f}{d\alpha^\nu} = -\sum p^{\nu k}\left(\frac{d^{\nu-1}}{du^{\nu-1}}\left(\frac{1}{e^u-1}\right)\right)_{u=\alpha p^k} = \sum_{r=1}^{\nu}\sum \frac{\gamma_\nu(r)\, p^{\nu k}}{(e^{\alpha p^k}-1)^r} = \sum_{r=1}^{\nu}\gamma_\nu(r)\sum\frac{p^{\nu k}}{(e^{\alpha p^k}-1)^r}$$

$$= \sum_{r=1}^{\nu}\gamma_\nu(r)\left\{\int_2^m \frac{x^{\nu k}dx}{(e^{\alpha x^k}-1)^r \log x} + O\left(e^{-c\log^{1/2} m}\alpha^{-\nu-\frac{1}{k}}\right)\right\}$$

$$= \int_2^m\left(\sum_{r=1}^{\nu}\frac{\gamma_\nu(r)\,x^{\nu k}}{(e^{\alpha x^k}-1)^r}\right)\frac{dx}{\log x} + E = -\int_2^m \frac{x^{\nu k}}{\log x}\left(\frac{d^{\nu-1}}{du^{\nu-1}}\left(\frac{1}{e^u-1}\right)\right)_{u=\alpha x^k} dx + E,$$

with $E = O\left(e^{-c\log^{1/2} m}\alpha^{-\nu-\frac{1}{k}}\right)$. The integral may be computed, along the pattern of [8], as follows:

$$\int_2^m \frac{x^{\nu k}}{\log x}\left(\frac{d^{\nu-1}}{du^{\nu-1}}\left(\frac{1}{e^u-1}\right)\right)_{u=\alpha x^k} dx = \alpha^{-\nu}\int_{2\alpha}^{\alpha m^k}\frac{k u^\nu}{\log(u/\alpha)}\frac{d^{\nu-1}}{du^{\nu-1}}\left(\frac{1}{e^u-1}\right)\frac{du}{(du/dx)}$$

$$= \alpha^{-\nu-\frac{1}{k}}\int_{2\alpha}^{\alpha m^k}\frac{u^{\nu-1+1/k}}{\log(u/\alpha)}\frac{d^{\nu-1}}{du^{\nu-1}}\left(\frac{1}{e^u-1}\right)du,$$

and

$$f_\nu = \frac{-1}{\nu!\,\alpha^{1/k}}\left(\int_{2\alpha}^{\alpha^{1/2}} + \int_{\alpha^{1/2}}^{\alpha^{-1/2}} + \int_{\alpha^{-1/2}}^{\alpha m^k}\right) + O(\alpha^\nu E).$$

For $u \geq 2\alpha$, $\log(u/\alpha) \geq \log 2$; hence, the first integral is $O(\alpha^{1/2k})$. The last one is $O(\alpha^{-(\nu-1+1/k)/2}e^{-\alpha^{-1/2}})$; hence,

$$\int_{2\alpha}^{\alpha m^k} = \int_{\alpha^{1/2}}^{\alpha^{-1/2}}\frac{u^{\nu-1+1/k}}{\log(u/\alpha)}\frac{d^{\nu-1}}{du^{\nu-1}}\left(\frac{1}{e^u-1}\right)du + O(\alpha^{1/2k}).$$

For $\alpha^{1/2} \leq u < \alpha^{-1/2}$, $\left|\frac{\log u}{\log \alpha}\right| < \frac{1}{2}$, so that the last integral equals

$$\frac{1}{\log(1/\alpha)}\int_{\alpha^{1/2}}^{\alpha^{-1/2}} u^{\nu-1+\frac{1}{k}}\frac{d^{\nu-1}}{du^{\nu-1}}\left(\frac{1}{e^u-1}\right)du$$

$$-\frac{1}{\log^2(1/\alpha)}\int_{\alpha^{1/2}}^{\alpha^{-1/2}} u^{\nu-1+\frac{1}{k}}\log u\,\frac{d^{\nu-1}}{du^{\nu-1}}\left(\frac{1}{e^u-1}\right)du + O(\log^{-3}(1/\alpha)).$$

The extension of the limits of integration to (o, ∞) introduced new errors, the largest of which, $O(\log^{-1}(1/\alpha)\cdot\alpha^{1/2k})$, is absorbed into $O(\alpha^{1/2k})$. Since

$$I_\nu = \int_o^\infty u^{\nu-1+\frac{1}{k}}\frac{d^{\nu-1}}{du^{\nu-1}}\left(\sum_{\mu=1}^{\infty} e^{-\mu u}\right)du = (-1)^{\nu-1}\zeta\left(1+\frac{1}{k}\right)\Gamma\left(\nu+\frac{1}{k}\right)$$

and integration by parts shows that $J_\nu = \int_0^\infty u^{\nu-1+\frac{1}{k}} \log u \frac{d^{\nu-1}}{du^{\nu-1}}\left(\frac{1}{e^u - 1}\right) du$ satisfies (22), one finally obtains the estimate

$$f_\nu = \frac{-1}{\nu!\alpha^{1/k}}\left(\frac{1}{\log(1/\alpha)} I_\nu - \frac{1}{\log^2(1/\alpha)} J_\nu + O(\log^{-3}(1/\alpha)) + O(\alpha^{1/2k})\right)$$

$$+ O\left(\alpha^{-1/k} e^{-c' \log^{1/2} n}\right),$$

whence assertion (iv) of Lemma 3 follows immediately.

COROLLARY 3. *There exist constants* c *and* C *such that*

$$c(n^\lambda \log^{-k} n)^{1/(k+1)} < |f_\nu| < C(n^\lambda \log n)^{1/(k+1)}.$$

Proof. If (1) or (1') holds, parts (i) and (ii) of Lemma 3 yield

$$c(n^\lambda \log^{-k} n)^{1/(k+1)} \le |f_\nu| \le C\nu^{-1}(n^\lambda \log^{-k} n)^{1/(k+1)};$$

if (2) holds, part (iii) of Lemma 3 gives the result.

12. PROOF OF THEOREM 2 (II)

By Corollary 3 and (23), for every $N > 3$ and $|\theta| \le \delta$,

$$\left|\sum_{N+1}^{\infty}\left(\frac{-i\theta}{\alpha}\right)^\nu f_\nu\right| = \left|\frac{\theta}{\alpha}\right|^N O\left\{\frac{\delta}{\alpha}(n^\lambda \log^{-k} m)^{1/(k+1)}\right\}$$

$$= \left|\frac{\theta}{\alpha}\right|^N f_N O\left(\frac{\delta}{\alpha}\log n\right) = \left|\frac{\theta}{\alpha}\right|^N f_N \cdot o(1),$$

so that $\Sigma_{\nu=3}^{\infty}(-i\theta/\alpha)^\nu f_\nu = \Sigma_{\nu=3}^{'\infty}(-i\theta/\alpha)^\nu f_\nu$, as in Section 10. Also,

$$|S| = \left|\sum_{\nu=3}^{N}{}'\left(\frac{-i\theta}{\alpha}\right)^\nu f_\nu\right| = \left|\frac{\theta}{\alpha}\right| O\left((n^\lambda \log n)^{1/(k+1)}\left(\frac{\delta}{\alpha}\right)^2\right),$$

so that

$$\sum_{\mu=Q+1}^{\infty} S^\mu/\mu! = \left|\frac{\theta}{\alpha}\right|^Q O\left(\left(\frac{\delta}{\alpha}\right)^{2Q+3}(n^\lambda \log n)^{(Q+1)/(k+1)}\right)$$

$$= \left|\frac{\theta}{\alpha}\right|^Q f_Q\, O\left(n^{-\lambda/2(k+1)} \log^\xi n\right),$$

where $\xi = 2(Q+1) - \frac{1}{2}(k+1)^{-1}$. Selecting $Q = N$, one obtains

$$D(\rho, \theta) = F(\rho)\exp\left(-\frac{1}{2}\theta^2 A_2\right)\left(\sum_{j=2}^{N^2/2}{}'(-1)^j\left(\frac{\theta}{\alpha}\right)^{2j} B_{2j} + f_1(\theta)\right),$$

where $f_1(\theta)$ is an odd function, B_{2j} is given by (18), and the dash indicates that f_N which enters in the definition of some B_{2j}'s has to be replaced by $f_N'' = f_N(1 + o(1))$. From this, (19) follows as seen in Section 10; this proves that $F(x)$ satisfies assumption (b) of the FL with $u(\rho)$ given by (20).

Concerning condition (c) of the FL, it is known (see (43) in [8]), that if $\theta_o = n^{-1}(n\alpha)^{(k+2)/(2k+3)}$ and $\delta_1 = 2\pi\theta_o$, then

$$\int_{|\theta|>\delta_1} |F(\rho e^{i\theta})|\, d\theta = F(\rho)\, O\left(n^{-1}\exp(-c(n\alpha)^{1/(2k+1)})\right),$$

so that

$$\int_{|\theta|>\delta_1} |F(\rho e^{i\theta})|\, d\theta = F(\rho)\, A_2^{-1/2} O(n^{-\omega}) \tag{24}$$

holds for every arbitrarily large ω. Since $\delta < \delta_1$ (this inequality is seen to be equivalent to $\gamma \log n < \pi^2 M^{1/(2k+3)}$), we still have to estimate $I = \int_\delta^{\delta_1} |F(\rho e^{i\theta})|\, d\theta$. By (13') and Corollary 3, for $|\theta| \leq \delta_1$,

$$\log|F(\rho e^{i\theta})| = \log F(\rho) - \frac{1}{2}\theta^2 A_2 + \psi(\theta),$$

with

$$\psi(\theta) = O\left(\left(\frac{\theta}{\alpha}\right)^4 (n \log n)^{1/(k+1)}\right) = \theta^2 A_2 \cdot O(\delta_1^2 \alpha^{-4} A_2^{-1} (n \log n)^{1/(k+1)}).$$

By Corollary 2, $A_2^{-1} = O(n^{-2+1/(k+1)}(\log n)^{-1+1/(k+1)})$, so that, by the definition of δ_1, $\psi(\theta) = \theta^2 A_2 \varepsilon$, with

$$\varepsilon = O(n^{-2/(2k+3)}(\log n)^{(4k+3)/(2k+3)})$$

and

$$|F(\rho e^{i\theta})| = F(\rho)\exp\left(-\frac{1}{2}\theta^2 A_2'\right), \qquad A_2' = A_2(1+\varepsilon).$$

Hence, integration gives

$$\int_\delta^{\delta_1} |F(\rho e^{i\theta})|\, d\theta = F(\rho)\int_\delta^{\delta_1} \exp\left(-\frac{1}{2}\theta^2 A_2'\right) d\theta = F(\rho)\, A_2'^{-1/2}\int_\zeta^{\zeta_1} e^{-u^2/2}\, du,$$

with $\zeta^2 = \delta^2 A_2' = 2\gamma(\log n)(1 + o(1))$, and the integral is $O(e^{-\zeta^2/2}/\zeta) = O(n^{-\gamma}\log^{-1/2} n)$. Consequently, by (24),

$$\int_{|\theta|>\delta} |F(\rho e^{i\theta})|\, d\theta = F(\rho)\, A_2^{-1/2} O(n^{-\gamma}\log^{-1/2} n),$$

and (c) of the FL holds for every τ if $\omega < \gamma$, and also for $\omega = \gamma$ if $\tau \geq -1/2$. This finishes the proof of Theorem 2.

13. PROOF OF THEOREM 1

Part (i) of Theorem 1 was proved in Section 6.

Proof of (ii). If (1) holds, then, by Lemma 3, $f_\nu = (-1)^\nu \nu^{-1} Mc_\nu$. Substitution in (18) yields $B_{2j} = \Sigma_{s=1}^{g} M^s v_{js} = O(M^g)$, as $M \to \infty$, for given j, with $g = [2j/3]$ and

$$v_{js} = \sum_{3,s,2j}^{N} \prod_{\mu=1}^{r} \frac{(c_{\nu_\mu} \nu_\mu^{-1})^{\alpha_\mu}}{\alpha_\mu !}.$$

Here, and in what follows, the letters above and below the summation sign stand for the conditions of summation

$$3 \leq \nu_1 < \nu_2 < \cdots < \nu_r \leq N, \qquad \sum_{\mu=1}^{r} \alpha_\mu = s, \qquad \sum_{\mu=1}^{r} \alpha_\mu \nu_\mu = 2j.$$

Omission of any letter below or above the Σ-sign means that the corresponding restriction has been dropped. If

$$w_{js} = \sum_{3,s,2j}^{N} \prod_{\mu=1}^{r} (\nu_\mu^{\alpha_\mu} \alpha_\mu !)^{-1},$$

then, by (21') and $\sum_{i=1}^{r} \alpha_\mu = s$, $v_{js} = w_{js}(1 + O(sn^{-4(1-\lambda)}))$. Also,

$$w_{js} \leq \sum_{3,s}^{N} \prod_{\mu=1}^{r} (\nu_\mu^{\alpha_\mu} \alpha_\mu !)^{-1} = \frac{1}{s!}\left(\frac{1}{3} + \cdots + \frac{1}{N}\right)^s = \frac{1}{s!}\left(\log N - 1 - \frac{1}{2} - \bar{\gamma} - \varepsilon\right)^s = \frac{a_1^s}{s!}.$$

Here $\bar{\gamma}$ stands for the Euler constant, $0 < \varepsilon < 1$, $\varepsilon \to 0$ as $N \to \infty$, and $a_1 = \log N_1$, $N_1 = N \exp\{-(3/2 + \bar{\gamma} + \varepsilon)\} < N/7$. Hence, $v_{js} = a_2^s/s!$ uniformly in j. Replacement of all the c_ν by ones leads to an error $O(s \cdot n^{-4(1-\lambda)} a_2^s/s!)$ on v_{js}, and to an error

$$O\left(n^{-4(1-\lambda)} \sum_{s=1}^{g} (Ma_2)^s/(s-1)!\right) = O(n^{-4(1-\lambda)} M^g)$$

on B_{2j}; hence,

$$B_{2j} = \sum_{s=1}^{g} M^s w_{js} + O(n^{-4(1-\lambda)} M^g).$$

Also, since $B_{2j} = O(M^g)$ and, by Corollary 2, $\alpha^2 A_2 = O(M)$ for $Q = N$, the error term in (19) becomes

$$O\left(n^{-\gamma} (\log n)^{-1/2} \sum_{j=2}^{N^2/2} M^{g-j} \frac{(2j)!}{2^j j!} (\log n)^j\right) = O(n^{-\gamma} M^{-1} (\log n)^{5/2})$$

$$= O(n^{-\gamma-\lambda/(k+1)} (\log n)^{\xi_1}),$$

where $\xi_1 = \frac{5}{2} - k/(k+1)$; hence, $\gamma_1 = \lambda/(k+1) > 0$ and $\xi = \xi_1$ in (19). If γ is selected arbitrarily, the FL and Theorem 2 show that the terms of $u(\rho)$ that are

$$O(n^{-\gamma} \log^{-1/2} n)$$

will be absorbed into the error term; hence, one keeps in $u(\rho)$ only terms with $M^{g-j} \geq n^{-\gamma} \log^{-1/2} n$, or $u(\rho) = \sum_{j=2}^{J} C_{2j} \alpha^{-2j} A_2^{-j} + O(n^{-\gamma} \log^{-1/2} n)$, with

$$C_{2j} = (-1)^j 1 \cdot 3 \cdots (2j-1) \sum_{s=1}^{g} M^s \sum_{3,s,2j} \prod_{\mu=1}^{r} \left(\nu_\mu^{\alpha_\mu} \alpha_\mu ! \right)^{-1}, \qquad J = [3\gamma(k+1)/\lambda].$$

Having selected some γ, and having determined J, take $N = 2J + 1$; then f_N (and, hence, f''_N), will not occur in any of the B_{2j} that are needed. Also, the restriction $\nu_r \leq N$ becomes vacuous, and it has been omitted. This finishes the proof of part (ii) of Theorem 1, with

$$(25) \qquad v_1(m, k) = \sum_{j=2}^{J} C_{2j} \alpha^{-2j} A_2^{-j}.$$

Remark. One may select γ arbitrarily large, so that the error term in (4') may be made arbitrarily small. But one cannot make $E_1 \to 0$ by letting $\gamma \to \infty$, because, if J is replaced by ∞, the series (25) diverges because of the factor $1 \cdot 3 \cdot 5 \cdots (2j-1)$ in C_{2j}.

Proof of (iii). From (14) with $\beta = \alpha$, it follows that

$$n\alpha = M - \frac{1}{2}\alpha \sum p^k - 2 \sum_{j=1}^{\infty} (-1)^j \left(\frac{\alpha}{2\pi}\right)^{2j} \zeta(2j) \sum p^{2kj}.$$

Adding this to (13), one obtains

$$\log F(\rho) + n\alpha = A_1 + n\alpha = M \log(e/\alpha) - k\theta(m) - E_2,$$

with $E_2 = \sum_{j=1}^{\infty} (-1)^j (2 - 1/j)(\alpha/2\pi)^{2j} \zeta(2j) \sum p^{2kj}$. If $\lambda \leq (2k+2)/(2k+3)$, then

$$E_2 = O(\alpha^2 \sum p^{2k}) = O(n^{-2+\lambda(2k+3)/(k+1)} (\log n)^{-k/(k+1)}).$$

Also, by Corollary 2, $A_2 = \alpha^{-2} M(1 + o(n^{-(2-\varepsilon)(1-\lambda)}))$. Introducing these into (12), with $\omega = \gamma$, $\tau = -1/2$, one obtains

$$(26) \qquad \begin{aligned} p(n, m; k) &= (2\pi M)^{-1/2} \alpha^{1-M} e^{M - k\theta(m)} (1 - E_2 + v_1(m; k) + E_4), \\ E_4 &= O(n^{-(2-\varepsilon)(1-\lambda)} + n^{-\gamma} \log^{-1/2} n). \end{aligned}$$

Since γ is arbitrarily large and $\varepsilon > 0$ is arbitrarily small, the dominant error term is E_2. Also, by (16') α has the value asserted in part (iii) of Theorem 1; this finishes the proof of part (iii).

(iv) If $\lambda > (k+1)/(k+2)$, then $v_1 = u(\rho) = O(M^{-1})$ is absorbed into E_2. If, however, $\lambda \leq (k+1)/(k+2)$, $E_2 = o(v_1)$, and, at the same time α can be expressed with sufficient precision in terms of n and m to lead to a true asymptotic formula. Replacing α in the second member of (26) by its value (16'), one obtains

$$n^{M-1} e^{-k\theta(m)}((2\pi)^{1/2} M^{M-1/2} e^{-M})^{-1}\left(1 + v_1(m; k) + \frac{1}{2}(k+1)^{-1} n^{-1} m^{k+2} \log^{-2} n + E_3\right),$$

all error terms of (26) being absorbed into

$$E_3 = O(n^{-1} m^{k+2} \log^{-3} m) = O(n^{-1+\lambda(k+2)/(k+1)} (\log n)^{-(2k+1)/(k+1)}).$$

If also $\alpha^2 A_2$ in (25) is replaced by $M(1+\varepsilon)$, where $\varepsilon = o(n^{-(2-\varepsilon)(1-\lambda)})$, this becomes

$$v_1(m; k) = \sum_{j=2}^{J} M^{-j} C_{2j}(1+\varepsilon) = \sum_{j=2}^{J} M^{-j} C_{2j} + E_5$$

$$= \sum_{j=2}^{J} (-1)^j M^{-j} \frac{(2j)!}{2^j j!} \sum_{s=1}^{g} M^s \sum_{3,s,2j} \prod_{i=1}^{r} \left(\nu_i^{\alpha_i} \alpha_i!\right)^{-1} + E_5 = \sum_{\mu=1}^{J-1} M^{-\mu} t_\mu + E_5,$$

with

$$t_\mu = (-1)^\mu \sum_{s=1}^{2\mu} (-1)^s \frac{(2s+2\mu)!}{2^{s+\mu}(s+\mu)!} \sum_{3,s,2(\mu+s)} \prod_{i=1}^{r} (\nu_i^{\alpha_i} \alpha_i!)^{-1}$$

and $E_5 = o(n^{-2+\lambda(2k+1)/(k+1)+\varepsilon} (\log n)^{k/(k+1)})$ for every $\varepsilon > 0$; this is

$$O(n^{-1+\lambda(k+2)/(k+1)} (\log n)^{-(2k+1)/(k+1)}).$$

If b_r denotes the Bernoulli numbers ($b_1 = -1/2$, $b_2 = 1/6$, $b_4 = -1/30$, $\cdots$; $b_{2j+1} = 0$ for $j > 0$) and

$$g(M) = \exp\left(-\sum_{j=1}^{(J+1)/2} M^{-2j+1} b_{2j}/2j(2j-1)\right)$$

is expanded in a power series $g(M) = \Sigma_{\nu=0}^{\infty} h_\nu M^{-\nu}$, one has $h_0 = 1$, $h_\nu = t_\nu$ for $1 \leq \nu \leq (J+1)/2$. (An indirect, but very simple proof for this expansion is obtained by equating the value of $p(n, m; 1)$ from (26), with $v_1 = \Sigma_{\mu=1}^{J-1} t_\mu M^{-\mu} + E_5$, to its value $\tilde{p}_1(n, m; 1)(1 + O(n^{-1}))$, obtained for $\lambda < 2/3$ in Section 7.) Hence, $1 + v_1(m; k) = g(M)(1 + O(M^{-J-2}))$ and, by Stirlings' formula for factorials, (26) now reads

$$\text{(26')} \qquad p(n, m; k) = \frac{n^{M-1} e^{-k\theta(m)}}{(M-1)!}\left(1 + \frac{1}{2}(k+1)^{-1} n^{-1} m^{k+2} \log^{-2} n + E_3\right).$$

Indeed, the last error term is $O(M^{-J-2})$, and it is absorbed into E_3 for

$$J \geq \frac{k+4}{\lambda}\left(\frac{k+1}{k+4} - \lambda\right).$$

This finishes the proof of (4'''), that is, of part (iv) of Theorem 1.

Proof of (v). If (2) holds, f_ν is given by (iv) of Lemma 3. Substitution of these values in (18) leads to

(18')
$$B_{2j} = \sum_{\mu=1}^{g} \phi_k^\mu \sum_{3,\mu,2j} \prod_{s=1}^{r} \psi_k(\nu_s), \qquad \phi_k = \zeta\left(1+\frac{1}{k}\right)\alpha^{-1/k}\log^{-1}\frac{1}{\alpha},$$
$$\psi_k(\nu_s) = \frac{1}{\alpha_s!}\left(\frac{\Gamma(\nu_s+1/k)}{\nu_s!}\left(1 - L_{\nu_s}\log^{-1}\frac{1}{\alpha} + O\left(\log^{-2}\frac{1}{\alpha}\right)\right)\right)^{\alpha_s}.$$

By Theorem 3 (see (16'''')), α depends now only on n; hence, on setting

$$u(\rho) = v_2(n;k) = \sum_{j=2}^{N^2/2} (-1)^j B_{2j}\alpha^{-2j}A_2^{-j}\frac{(2j)!}{2^j j!} + E_6, \tag{27}$$

with B_{2j} defined by (18') and $E_6 = O(n^{-\gamma-1/(k+1)}(\log n)^{(5/2)+k/(k+1)})$, assertion (v) of Theorem 1 is proved. The estimate for the error term follows from (19) and (with a slight anticipation) an estimate of the individual terms in (27).

14. EVALUATION OF $u(\rho)$

If a larger error term is accepted, the unwieldy expression for $v_2(n;k)$ can be much simplified. Since $\psi_k(\nu_s) = O(1)$ and $\phi_k = O((n\log^{-k} n)^{1/(k+1)})$,

$$B_{2j} = H_{2j}\,\phi_k^g(1 + O((n^{-1}\log^k n)^{1/(k+1)})),$$

where the H_{2j} are numerical constants, depending only on j and k. Also, by Theorem 3 and part (ii) of Corollary 2, $\alpha^2 A_2 = O((n\log^{-k} n)^{1/(k+1)})$. Hence, in (27) the individual terms are $O((n\log^{-k} n)^{(g-j)/(k+1)})$, the largest being the first two, corresponding to $j = 2$ and $j = 3$, which are both $O((n^{-1}\log^k n)^{1/(k+1)})$. We observe that B_4 contains only one term. If each of the B_{2j}'s is replaced by its largest term, it now follows that one introduces into (27) an additional error $O((n^{-1}\log^k n)^{2/(k+1)})$. However, already the third term ($j = 4$, $g = 2$) is only of that order and is absorbed into the error term. Retaining, therefore, in (27) only the first two terms $1\cdot 3B_4\alpha^{-4}A_2^{-2} - 1\cdot 3\cdot 5B_6\alpha^{-6}A_2^{-3}$, and replacing B_4 and B_6 by their values (18'), one obtains

$$v_2(n;k) = 3\left\{\Gamma\left(4+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\left(1 - L_4\log^{-1}\frac{1}{\alpha} + O\left(\log^{-2}\frac{1}{\alpha}\right)\right)\alpha^{-1/k}\left(\log^{-1}\frac{1}{\alpha}\right)\Big/4!\right\}\alpha^{-4}A_2^{-2}$$
$$- (3\cdot 5/2!)\left\{\Gamma\left(3+\frac{1}{k}\right)\zeta\left(1+\frac{1}{k}\right)\left(1 - L_3\log^{-1}\frac{1}{\alpha}\right.\right.$$
$$\left.\left. + O\left(\log^{-2}\frac{1}{\alpha}\right)\right)\alpha^{-1/k}\left(\log^{-1}\frac{1}{\alpha}\right)\Big/3!\right\}^2\cdot\alpha^{-6}A_2^{-3} + E_7$$

with $E_7 = O(n^{-1}\log^k n)^{2/(k+1)}$. Finally, replacing here α and A_2 by their values from (16'''') and Corollary 2, one obtains

$$v_2(n;k) = \frac{1}{8}\frac{k}{k+1}\Gamma\left(4+\frac{1}{k}\right)\left\{\left(\zeta\left(1+\frac{1}{k}\right)\right)^k\left(\Gamma\left(2+\frac{1}{k}\right)\right)^{2k+1} n\log^{-k} n\right\}^{-1/(k+1)}$$

(27')
$$-\frac{5}{24}\frac{k}{k+1}\Gamma^2\left(3+\frac{1}{k}\right)\left\{\left(\zeta\left(1+\frac{1}{k}\right)\right)^k\left(\Gamma\left(2+\frac{1}{k}\right)\right)^{3k+2} n\log^{-k} n\right\}^{-1/(k+1)}$$

$$+ E_8,$$

$$E_8 = O\left(\frac{\log\log n}{(n\log n)^{1/(k+1)}}\right);$$

this finishes the proof of Theorem 1.

15. THE CASE k = 1

In case $k = 1$, some of the preceding results are simpler. We summarize them as follows:

THEOREM 4. (i) *If* m *is constant, then*

$$p(n,m) = \frac{n^{M-1}e^{-\theta(m)}}{(M-1)!}\{1+\psi(n)\},$$

where $\psi(n)$ *is* $O(n^{-1})$ *and can be determined explicitly.*

(ii) *If* $m^2 = n^\lambda \log m$, $\lambda < 1$, *then* $p(n, m) = p_1(n, m; 1)(1 + v_1(m) + E_9)$, *where* $p_1(n, m; 1)$ *is defined by* (12),

$$\alpha = \frac{M}{n}\left\{1 - \frac{m^2}{4n\log m}(1+O(\log^{-1} m))\right\},$$

$$n\alpha + A_1 = M\log\frac{e}{\alpha} - \theta(m) - \sum_{j=1}^{\infty}(-1)^j\left(2-\frac{1}{j}\right)\left(\frac{\alpha}{2\pi}\right)^{2j}\zeta(2j)\sum p^{2j},$$

$A_2 = M^{-1}n^2(1 + O(n^{-1+\lambda}))$, $v_1(m) = v_1(m; 1)$ *as given by* (25), *and* $E_9 = O(n^{-\gamma}\log^{-1/2} n)$ ($\gamma > 0$, *arbitrary*).

(iii) *If, furthermore,* $\lambda \leq 4/5$, *then*

$$p(n,m) = (2\pi M)^{-1/2}\alpha^{1-M}\exp\{M-\theta(m)\}(1+v_1(m)+E_{10}),$$

with α *and* $v_1(m)$ *as under* (ii), *and* $E_{10} = O(n^{-2+5\lambda/2}\log^{-1/2} n)$.

(iv) *If, furthermore,* $\lambda \leq 2/3$, *then*

$$p(n,m) = \frac{n^{M-1}e^{-\theta(m)}}{(M-1)!}\left(1+\frac{m^3}{4n\log^2 m}+E_{11}\right),$$

with $E_{11} = O(n^{-1+3\lambda/2}\log^{-3/2} n)$.

(v) *If* $m^2 = n\log^\mu n$, $\mu > 1$, *then*

$$p(n,m) = p(n) = p_1(n,m;1)(1+v_2(n)+E_{12}),$$

with $p_1(n, m; 1)$ *defined by* (12), $v_2(n) = v_2(n; 1)$ *given by* (27), *and*

$$E_{12} = O(n^{-\gamma} \log^{-1/2} n)$$

($0 < \gamma < \infty$, *arbitrary*).

(vi) $v_2(n) = -\frac{3\sqrt{3}}{16\pi}\left(\frac{\log n}{n}\right)^{1/2}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right)$.

Theorem 4 follows almost trivially from Theorem 1 and its proof, on setting everywhere $k = 1$.

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