THE FOURIER COEFFICIENTS OF AUTOMORPHIC FORMS ON HOROCYCLIC GROUPS, III

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1. INTRODUCTION

In previous papers, we have shown how the circle method can be used to determine the Fourier coefficients of *entire* automorphic forms on certain horocyclic groups *(Grenzkreisgruppen)*. In the present paper, we extend the results to automorphic forms that have poles.

Our results are contained in the following theorems. For definitions of symbols, see [2], except as noted; automorphic forms with poles are defined in Section 2 of the present paper.

THEOREM 1. *Let* $F(z)$ *be an automorphic form of dimension* $r > 0$ *on an* H-*group* Γ. *Let* R_0 *be a fixed fundamental region of* Γ *which does not have* ∞ *as a vertex, and let* $p_1, p_2, \cdots, p_s$ *be a complete set of inequivalent vertices of* Γ. *Let the expansions of* $F(z)$ *at the parabolic points be*

$$(1.1)\qquad \begin{aligned} F(z) &= (A_k z)^{-r} t_k^{\alpha_k} f_k(t_k) \qquad (t_k = e(A_k z/\lambda_k)), \\ f_k(t) &= \sum_{m=-\mu_k}^{\infty} a_m^{(k)} t^m \qquad (1 \leq k \leq s). \end{aligned}$$

Let $F(z)$ *have simple poles at the interior points* $z_1, z_2, \cdots, z_q$ *of* R_0 *with residues* $B_1, \cdots, B_q$. *Then, for each* k $(1 \leq k \leq s)$, *the Fourier coefficients* $a_m^{(k)}$ *with* $m \geq 0$ *are given in terms of the set of coefficients* $a_m^{(j)}$ *with* $m < 0$ $(1 \leq j \leq s)$ *by the formula*

$$(1.2)\qquad \begin{aligned} a_m^{(k)} &= \left(\frac{2\pi}{\lambda_k}\right) e(r/4) \sum_{j=1}^{s} \sum_{\nu=1}^{\mu_j} a_{-\nu}^{(j)} \sum_{c_{jk} \in C'_{jk}} c_{jk}^{-1} A(c_{jk}, \nu_j, m_k)\, L(c_{jk}, m_k, \nu_j, r) \\ &- \left(\frac{2\pi i}{\lambda_k}\right) \sum_{n=1}^{q} B_n \sum_{V \in \Delta_k(z_n)} \varepsilon(V)\, e(-(m + \alpha_k) A_k V z_n/\lambda_k)\, (A_k V z_n)^{r+2} (c z_n + d)^{-r-2}. \end{aligned}$$

Here C'_{jk} *is the set of positive elements of* C_{jk}, *and* Δ_k *is defined in (4.5).*

Poles of higher order can be treated in an analogous manner, but the algebraic details, into which we do not enter here, become rather complicated.

THEOREM 2. *If* $F(z)$ *is an automorphic form of dimension zero on* Γ *having the expansions (1.1), then, for* $m \geq 1$, $a_m^{(k)}$ *is given by the series (1.2) with an error term* $O(1)$ $(m \to \infty)$, *where in the first series* c_{jk} *is further restricted by the*

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condition $c_{jk} < \beta\sqrt{m/h}$ (β = constant), *and in the second series we impose the additional requirement* $\Im A_k V z_n > 1/\beta^2 m$ $(1 \le n \le q)$. *Here* h *is defined in (2.17).*

THEOREM 3. *Let* G(z) *be an automorphic form of dimension* -2 *on* Γ *with Fourier coefficients* $b_m^{(k)}$ *which is, moreover, the derivative of a form* F(z) *(of dimension zero). Then, for* $m \geq 1$, $b_m^{(k)}$ *is equal to* $2\pi i(m + \alpha_k)/\lambda_k$ *times the* $a_m^{(k)}$ *of Theorem 2, plus an error term* O(m).

Theorem 3 is, of course, obtained from Theorem 2 by differentiating (1.1).

A special case of Theorem 1 was proved by Zuckerman [5]. He considered automorphic forms on the modular group, which has a single parabolic cusp (s = 1, in our notation). He solved a slightly different problem which, however, is equivalent to the one treated in this paper. For the path of integration, he used a variant of the Farey series.

Petersson has proved Theorem 1 by another method [3].

2. GROUPS AND AUTOMORPHIC FORMS

In this section, we recall needed material from [2, Sections 2, 3, and 4]. We also define automorphic forms, which may now have poles, and develop some of their properties.

Let Γ be an H-group [2, Section 2]. Let p_k $(k = 0, 1, 2, \cdots)$ be a parabolic vertex of Γ with $p_0 = \infty$. Let R_0 be a fundamental region of Γ which does not have ∞ as a vertex. We denote by $p_1, p_2, \cdots, p_s$ a complete set of inequivalent parabolic vertices of R_0. Define

$$A_k = \begin{pmatrix} 0 & -1 \\ 1 & -p_k \end{pmatrix} \quad (p_k \neq \infty); \qquad A_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.1}$$

so that $A_k(p_k) = \infty$. Consider the set

$$A_j \Gamma A_k^{-1} = \{X \mid X = A_j V A_k^{-1},\ V \in \Gamma\} \qquad (j, k = 0, 1, \cdots, s),$$

and write

$$X = \begin{pmatrix} a_{jk} & b_{jk} \\ c_{jk} & d_{jk} \end{pmatrix}. \tag{2.2}$$

Call C_{jk} the set of third entries c_{jk} of $A_j \Gamma A_k^{-1}$:

$$C_{jk} = \left\{ x \mid \exists \begin{pmatrix} \cdot & \cdot \\ x & \cdot \end{pmatrix} \in A_j \Gamma A_k^{-1} \right\} \qquad (j, k = 0, 1, \cdots, s), \tag{2.3}$$

and call C'_{jk} the set of positive elements of C_{jk}:

$$C'_{jk} = \{c_{jk} \in C_{jk} \mid c_{jk} > 0\}. \tag{2.4}$$

The set of real numbers C_{jk} *is discrete.* (This was proved in [2, Section 2] for the case $j, k \neq 0$, but the proof works for all j, k.) Define

$$\overline{c_{jk}} = \min c_{jk} \qquad (c_{jk} \in C'_{jk}). \tag{2.5}$$

Then

(2.6) $$\overline{c_{jk}} > 0 \qquad (j, k = 0, 1, \cdots, s).$$

Corresponding to each p_k, there is a subgroup Γ_k of Γ that leaves p_k fixed. There exists a positive number λ_k such that the cyclic group Γ_k is generated by

(2.7) $$S_k = \begin{pmatrix} 1 + \lambda_k p_k & -\lambda_k p_k^2 \\ \lambda_k & 1 - \lambda_k p_k \end{pmatrix} \quad (k > 0), \qquad S_0 = \begin{pmatrix} 1 & -\lambda_0 \\ 0 & 0 \end{pmatrix}.$$

We note that

(2.8) $$S_k = A_k^{-1} U^{-\lambda_k} A_k, \qquad S_k^m = A_k^{-1} U^{-m\lambda_k} A_k \qquad (k = 0, 1, 2, \cdots),$$

where m is an integer, $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and we write, symbolically, $U^\kappa = \begin{pmatrix} 1 & \kappa \\ 0 & 1 \end{pmatrix}$ for real κ.

We proceed to the definition of an automorphic form. Let z be a complex variable and $\mathscr{H}$ the upper half-plane $\Im z > 0$. Write $e(z) = \exp 2\pi i z$. By an admissible multiplier system $\varepsilon(\Gamma, r)$ for the group Γ and the (real) dimension r, we shall mean a complex-valued function $\varepsilon(V)$ defined for $V \in \Gamma$ such that

1) $|\varepsilon(V)| = 1$,

2) $\varepsilon(V_1 V_2)/\varepsilon(V_1)\varepsilon(V_2) = m(V_1, V_2 z)\, m(V_2, z)/m(V_1 V_2, z)$

for $z \in \mathscr{H}$ and $V_1, V_2 \in \Gamma$. Here $m(V, z) = (cz + d)^{-r}$ for $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and the branch of the many-valued function is specified in [2, (1.2)] (note that if it were not for many-valuedness, the right-hand member of 2) would be identically 1).

We now define an automorphic form of (real) dimension r on Γ to be a function $F(z)$ such that

1) $F(z)$ is meromorphic in $\mathscr{H}$,

2) $F(z)$ tends to a definite limit, finite or infinite, as z tends to a parabolic vertex from within a fundamental region of Γ,

(2.9) 3) $F(Vz) = \varepsilon(V)\, m(V, z) f(z)$ for each $V \in \Gamma$.

We now show that these conditions lead directly to the existence of an expansion for $F(z)$ around a parabolic vertex.

LEMMA 1. *Corresponding to each finite parabolic vertex* p_k $(k > 0)$, *there exists a number* $\tau_k > 0$ *such that*

(2.10) $$(A_k z)^r t_k^{-\alpha_k} F(z) = f_k(t_k) \qquad (t_k = e(A_k z/\lambda_k)),$$

(2.11) $$f_k(t) = \sum_{m=-\mu_k}^{\infty} a_m^{(k)} t^m \qquad (\mu_k \text{ finite}),$$

the series converging for $|t| < \tau_k$.

Proof. We first show that $f_k(t)$ is single-valued. If $t = e(A_k z_1/\lambda_k) = e(A_k z_2/\lambda_k)$, then $A_k z_1 = A_k z_2 + m\lambda_k$ (m an integer). Hence $A_k z_1 = U^{m\lambda_k} A_k z_2 = A_k S_k^{-m} z_2$, or

$z_1 = S_k^{-m} z_2$ (see (2.8)). Denoting the left member of (2.10) by $\psi(z)$, we have

$$\psi(z_2) = (A_k S_k^m z_1)^r e(-\alpha_k A_k S_k^m z_1/\lambda_k) F(S_k^m z_1)$$

$$= (U^{-m\lambda_k} A_k z_1)^r e(-\alpha_k U^{-m\lambda_k} A_k z_1/\lambda_k) \varepsilon(S_k^m) m(S_k^m, z_1) F(z_1),$$

where we have used (2.9). But

$$e(-\alpha_k) = \varepsilon(S_k) \quad [2, (3.2)],$$

(2.12)

$$m(S_k^m, z_1) = ((A_k z_1 - m\lambda_k)/A_k z_1)^{-r} = (U^{-m\lambda_k} A_k z_1/A_k z_1)^{-r}.$$

Hence,

$$\psi(z_2) = e(\alpha_k m) e(-\alpha_k A_k z_1/\lambda_k) \varepsilon(S_k^m) (A_k z_1)^r F(z_1)$$

$$= (A_k z_1)^r e(-\alpha_k A_k z_1/\lambda_k) F(z_1) = \psi(z_1).$$

Thus $f_k(t)$ has the same value regardless of what solution z of $t = e(A_k z/\lambda_k)$ we choose.

From (2.10) it is clear that $f_k(t)$ is regular at $t = t^* \neq 0$ if and only if $F(z)$ is regular at $z = z^* \neq p_k$, where $t^* = (A_k z^*/\lambda_k)$. Suppose that $F(z)$ has a pole at $z^* \in \mathscr{H}$; then $|F(z)| \to \infty$ as $z \to z^*$. This is possible if and only if $|f_k(t)| \to \infty$ as $t \to t^*$; hence, $f_k(t)$ has a pole at t^*. The function $f_k(t)$, then, is meromorphic in $|t| < 1$ except possibly at $t = 0$, and the poles of $f_k(t)$ correspond in a one-to-one way with the poles of $F(z)$ in a fixed fundamental region R having the parabolic vertex p_k.

Suppose $f_k(t)$ has infinitely many poles in a neighborhood of $t = 0$. Then $F(z)$ has infinitely many poles z_n in a neighborhood of p_k, and, on a certain subsequence, $z_n \to p_k$. Hence, $|F(z_n)| \to \infty$ as $z_n \to p_k$. Moreover, $f_k(t)$ approaches every value as $t \to 0$. Consider a sequence $\{v_n\}$ $(v_n \to 0)$ on which $f_k(t) \to 0$. According to (2.10), $F(w_n) \to 0$, where $v_n = (A_k w_n/\lambda_k)$, $w_n \in R$. Clearly $w_n \to p_k$ as $n \to \infty$. Thus, on the sequences $\{z_n\}$ and $\{w_n\}$, which both approach p_k, $F(z)$ has different limiting values, which violates the second condition of the definition of an automorphic form.

The argument shows not only that $f_k(t)$ has only a finite number of poles in $|t| < 1$ (and therefore that $F(z)$ has at most finitely many poles in a fundamental region), but also that f_k is regular or has a pole at $t = 0$. This proves (2.11).

Remark. The preceding reasoning justifies the statement made in [2], lines following (1.1): "The last condition is equivalent to the following: $F(z)$ has at most a pole (not an essential singularity) in the local variable appropriate to a given parabolic point."

At the parabolic point ∞, the discussion is similar, but we find it convenient to use the definition

$$e(\alpha_0) = \varepsilon(S_0) e(-r) \quad (0 \leq \alpha_0 < 1) \tag{2.13}$$

instead of (2.12). Then we find easily that

(2.14) $$F(z) = e(\alpha_0 z/\lambda_0) f_0(t_0) \qquad \left(f_0(t) = \sum_{m=-\mu_0}^{\infty} a_m^{(0)} t^m, \quad t_0 = e(z/\lambda_0) \right),$$

the series converging for $|t| < \tau_0$ $(\tau_0 > 0)$.

The transformation formula (2.9) for $F(z)$, when expressed in terms of the functions f_k, takes the form:

(2.15) $$f_k(e(w/\lambda_k)) = \varepsilon^{-1}(V)\, m^{-1}(A_j V A_k^{-1}, w)\, e(\alpha_j w'/\lambda_j - \alpha_k w/\lambda_k) f_j(e(w'/\lambda_j))$$
$$(w' = A_j V A_k^{-1} w;\ j, k = 1, 2, \cdots),$$

provided $A_j V A_k^{-1}$ does not have ∞ as a fixed point. (See [2, (3.6)].) If $j = 0$, we get

(2.16) $$f_k(e(w/\lambda_k)) = \eta \cdot \varepsilon^{-1}(V)\, m^{-1}(V A_k^{-1}, w)\, e(\alpha_0 w'/\lambda_0 - \alpha_k w/\lambda_k) f_0(e(w'/\lambda_0))$$
$$(w' = V A_k^{-1} w,\ k = 1, 2, \cdots),$$

where $|\eta| = 1$.

Let R_0 be a fixed fundamental region of Γ which does not have ∞ as a vertex. For purposes of integration, we shall use a partition of the segment $L(N, k) = L_N$: $0 \le x < \lambda_k$, $y = N^{-2}$, similar to the one employed in [2, Section 4]. Define

$$I_{jk}(V, N) = I_{jk}(V) = \{w \in L_N \mid A_j V A_k^{-1} w \in \operatorname{Int} E\} \qquad (w = x + iy),$$

(2.17) $$E = \{z \mid \Im z \ge h\},$$

$$h = \max_{1 \le j,k \le s} (1/\overline{c_{jk}},\ -(2\pi)^{-1} \lambda_j \log \tau_j/2).$$

The proof in [2] that the $I_{jk}(V)$ do not meet in interior points applies word-for-word to the present case, since the essential property used in that proof was the inequality $h \ge 1/\overline{c_{jk}}$ for $1 \le j, k \le s$. The set $M_{jk}(N)$ of elements V of Γ on which I_{jk} is not empty can be characterized by

(2.18) $$M_{jk}(N) = \{V \in \Gamma \mid 0 < c_{jk} < Nh^{-1/2},\ -\kappa/N \le -d_{jk}/c_{jk} < \lambda_k + \kappa/N,\ 0 \le a_{jk}/c_{jk} < \lambda_j\}$$
$$(j, k = 1, 2, \cdots, s),$$

where $\kappa = (1/\overline{c_{jk}}^{\,2} h - N^{-2})^{1/2}$, and where $a_{jk}, \cdots$ are defined in (2.2).

Let

$$I_0 = L_N - \bigcup_{j=1}^{s} \bigcup_{V \in M_{jk}} I_{jk}(V).$$

Define D_0 to be the part of R_0 exterior to the discs D_j $(1 \le j \le t)$ of diameter h^{-1} tangent to the real axis at p_j, where $p_1, p_2, \cdots, p_t$ is a *complete* set of parabolic vertices of R_0. *The region* D_0 *lies between two horizontal lines at heights* h_0, h_1 $(h_0 > h_1)$ *above the real axis.*

Let

(2.19) $$I_{0,k}(V) = \{w \in L_N \mid VA_k^{-1} w \in D_0\} .$$

We have

(2.20) $$L_N = \bigcup_{j=0}^{s} \bigcup_{V \in M_{jk}} I_{jk}(N),$$

$M_{0,k}(N)$ being the set of V on which $I_{0,k}(V)$ is not empty. This is the desired partition of L_N into nonoverlapping sets. An immediate consequence is that, for each N,

(2.21) $$\sum_{j=0}^{s} \sum_{V \in M_{jk}(N)} |I_{jk}(V)| = \lambda_k \quad (1 \le k \le s),$$

where $|I_{jk}|$ denotes the measure of I_{jk}.

3. INTEGRATION

Our object is to determine the coefficients $a_m^{(k)}$ $(m \ge 0)$ of (1.1) from a knowledge of the principal parts of F at the full set of parabolic vertices, that is, of the $a_m^{(j)}$ $(m < 0,\ 1 \le j \le s)$, together with the principal parts at the singularities of F. Since $f_k(t)$ is regular for $0 < |t| < \tau_k$, we have, by Cauchy's theorem,

$$a_m^{(k)} = \frac{1}{2\pi i} \int_K f_k(t)\, t^{-m-1} dt,$$

where K is the circle of radius $\tau_k/2$ about the origin. Making the change of variable $t = e(w/\lambda_k)$ $(w = x + iy)$, we get

$$\lambda_k a_m^{(k)} = \int_{L'} f_k(e(w/\lambda_k))\, e(-mw/\lambda_k)\, dw,$$

L' being the segment $0 \le x < \lambda_k$, $y = y_2 = -(2\pi)^{-1} \lambda_k \log(\tau_k/2)$. We now shift the path of integration downwards to the segment L_N: $0 \le x < \lambda_k$, $y = y_1 = N^{-2}$. The contributions to the integral from the vertical sides of the rectangle $(x = 0, \lambda_k;$ $y_1 < y < y_2)$ cancel, for the integrand is periodic with period λ_k. Hence,

(3.1) $$\lambda_k a_m^{(k)} = \int_{L_N} f_k(e(w/\lambda_k))\, e(-mw/\lambda_k)\, dw - 2\pi i \sum_{G(N)} \mathrm{Res} = P_1(N) - P_2(N),$$

where $G(N) = G_k(N)$ is the rectangle bounded by L', L_N, and the vertical sides just mentioned.

To treat the term $P_1(N)$, we make use of the dissection of L_N explained in Section 2, namely (2.20). The integral over L_N is then a sum of integrals over the sets $I_{jk}(V)$. In each of these integrals, we apply the transformation equation (2.15) or (2.16), according as $j > 0$ or $j = 0$, and then replace f_j $(j > 0)$ by the series (2.11). The result is

$$P_1(N) = \sum_{j=1}^{s} \varepsilon^{-1}(V) \sum_{V\in M_{jk}} \int_{I_{jk}(V)} m^{-1}(A_j V A_k^{-1}, w) \sum_{n=-\mu_j}^{\infty} a_n^{(j)} e\{(n+\alpha_j)w'/\lambda_j - m_k w\}\, dw$$

$$(3.2) \qquad + \eta \sum_{V\in M_{0,k}} \varepsilon^{-1}(V) \int_{I_{0,k}(V)} m^{-1}(VA_k^{-1}, w)\, e(\alpha_0 w'/\lambda_0 - m_k w)\, f_0(e(w'/\lambda_0))\, dw$$

$$= T_1(N) + T_2(N),$$

where α_j is defined in (2.12) and $m_k = (m + \alpha_k)/\lambda_k$.

The structure of $T_1(N)$ is exactly the same as that of the sum T_1 in [2, (5.4)]. In fact, the only difference in the two terms lies in the value of h; compare (2.17) and [2, (4.10)]. An examination of the argument of [2] reveals the fact that only two properties of h were used in the evaluation of T_1, namely,

1) $h \geq 1/c_{jk} \quad (1 \leq j, k \leq s)$,

2) the series $\sum_n |a_n^{(j)}| \exp(-2\pi(n + \alpha_j)h/\lambda_j)$ converges.

In the present case, 1) holds by (2.17). Moreover,

$$\exp(-2\pi(n + \alpha_j)h/\lambda_j) \leq \exp((n+1)\log(\tau_j/2)) = (\tau_j/2)^{n+1},$$

again by (2.17), and $\Sigma\, |a_n^{(j)}|\,(\tau_j/2)^{n+1}$ converges (see Lemma 1). Hence, both requirements on h are met, and we can take over the value of T_1 given in the lines following [2, (6.3)]:

$$(3.3) \quad T_1 = \lim_{N\to\infty} T_1(N) = 2\pi\, e(r/4) \sum_{j=1} \sum_{\nu=1}^{\mu_j} a_{-\nu}^{(j)} \sum_{c_{jk}\in C_{jk}'} c_{jk}^{-1} A(c_{jk}, \nu_j, m_k)\, L(c_{jk}, m_k, \nu_j, r).$$

(The new symbols in (3.3) are defined in [2, Section 1].) The absolute convergence of the series follows from estimates in [2, Section 7].

We now take up $T_2(N)$. As we saw in Section 2, the image of $I_{0,k}(V)$ under VA_k^{-1} lies in the region D_0, and D_0 lies between two horizontal lines at heights h_0, h_1 ($h_0 > h_1$) above the real axis. However, the image path might pass through one or more poles of the integrand. Since the integrand equals a regular function times $F(w')$, where $w' = VA_k^{-1}w$, we are here concerned with the poles of $F(w')$. Only a finite number of such poles lie in D_0, since D_0 is compact. Surround each pole w_j' ($j = 1, 2, \cdots, q$) in D_0 by an open disc C_j' of radius ε' small enough so that these discs lie entirely in D_0 and do not intersect. The quantity ε' does not depend on N.

If the image path $J = VA_k^{-1}(I_{0,k}(V))$ meets C_1', say, denote by ω_1' the intersection of J and C_1', and by C_1 the inverse image of C_1'. The point ω_1' divides the boundary of C_1' into two arcs; let K_1' be that arc whose inverse image is the smaller arc of C_1. Call ω_1 and K_1 the inverse images of ω_1' and K_1'; ω_1 is then an interval in L_N, while K_1 is a circular detour connecting the endpoints of ω_1. The set obtained from $I_{0,k}(V)$ by replacing all ω's by K's will be called $I_{0,k}'(V)$.

Since K_1 is not more than a semicircle, we have $|K_1|/|\omega_1| \leq \pi/2$. This shows that $|I_{0,k}'(V)| \leq 2^{-1}\pi\, |I_{0,k}(V)|$; hence,

$$\sum_{V\in M_{0,k}} |I'_{0,k}(V)| \le 2^{-1}\pi \sum_{V\in M_{0,k}} |I_{0,k}(V)| \le 2^{-1}\pi\lambda_k = C, \tag{3.4}$$

where we have used (2.21), and where C denotes a general constant which is independent of N but may depend on m or on any of the other parameters.

Moreover, if K_1 lies below L_N, then $\Im w \le N^{-2}$ when $w \in I'_{0,k}(V)$. Suppose that K_1 lies above L_N. Since C_1 lies entirely in $\mathscr{H}$, the radius ε of C_1 is less than N^{-2}. Hence, when w is on K_1, $\Im w \le N^{-2} + \varepsilon \le 2N^{-2}$. In all cases, then, we have

$$\Im w \le 2N^{-2} \qquad (w \in I'_{0,k}(V)).$$

We continue with the estimation of $T_2(N)$. First, replace the paths $I_{0,k}(V)$ by $I'_{0,k}(V)$. When $w\in I'_{0,k}(V)$, $w' = VA_k^{-1} w$ lies, independently of N, in a fixed compact subregion of D_0 which is free of singularities of F, namely, the part of D_0 exterior to the discs $C'_1, \cdots, C'_q$. Hence,

$$|F(w')| \le C \qquad (w\in I'_{0,k}(V)).$$

Also, we have $\Im w \le 2N^{-2}$, and since $w' \in D_0$, it follows that

$$|m^{-1}(VA_k^{-1}, w)|^2 = |\gamma w + \delta|^{2r} = |\Im w/\Im w'|^r \le (2N^{-2}h_1^{-1})^r \le CN^{-2r}.$$

Now the integrand of $T_2(N)$ equals $m^{-1}(VA_k^{-1}, w)\, e(-m_k w)\, F(w')$ (see (3.2) and (2.14)). The above estimates then give

$$|T_2(N)| \le \sum_{V\in M_{0,k}} |I'_{0,k}(V)|\, CN^{-r} \exp(Cm/N^2) \le CN^{-r},$$

in view of (3.4). It follows that

$$T_2 = \lim_{N\to\infty} T_2(N) = 0. \tag{3.5}$$

Combining (3.2), (3.3), and (3.5), we get

$$P_1 = \lim_{N\to\infty} P_1(N) = 2\pi\, e(r/4) \sum_{j=1}^{s} \sum_{=1}^{\mu_j} a_{-\nu}^{(j)} \sum_{c_{jk}\in C'_{jk}} c_{jk}^{-1} A(c_{jk}, \nu_j, m_k)\, L(c_{jk}, m_k, \nu_j, r). \tag{3.6}$$

4. THE SUM OF THE RESIDUES

We now have to treat the term $P_2(N)$ of (3.1), that is, $2\pi i$ times the sum of the residues of $f_k(e(w/\lambda_k))\, e(-mw/\lambda_k)$ in the rectangle $G_k(N)$. The limit of $P_2(N)$ as $N \to \infty$ is then the series

$$P_2 = 2\pi i \sum_{G_k} \operatorname{Res} f_k(e(w/\lambda_k))\, e(-mw/\lambda_k), \tag{4.1}$$

where G_k is the region $0 \le \Re w < \lambda_k$, $\Im w < (-2\pi)^{-1}\lambda_k \log(\tau_k/2)$. The series converges by virtue of (3.1) and of the existence of $\lim P_1(N) = P_1$.

We shall evaluate P_2 under the assumption that F has a simple pole of residue 1 at an interior point of the fundamental region R_0 and is otherwise regular in R_0. Let this pole be located at z_0; then F has simple poles at Vz_0 $(V \in \Gamma)$. Now, by (2.10),

$$f_k(e(w/\lambda_k))\, e(-mw/\lambda_k) = w^r e(-(m + \alpha_k)w/\lambda_k)\, F(A_k^{-1} w).$$

This expression is therefore singular when $A_k^{-1} w = Vz_0$, and its residue is

$$(A_k Vz_0)^r e(-(m + \alpha_k) A_k Vz_0/\lambda_k) \operatorname*{Res}_{w=A_k Vz_0} F(A_k^{-1} w). \tag{4.2}$$

To calculate the residue of F, we write

$$\begin{aligned}\operatorname*{Res}_{w=A_kVz_0} F(A_k^{-1} w) &= \lim_{z\to z_0} (A_k Vz - A_k Vz_0)\, F(Vz) \\ &= \lim_{z\to z_0} (A_k Vz - A_k Vz_0)\, \varepsilon(V)\,(cz + d)^{-r} F(z) \\ &= \varepsilon(V)\,(cz_0 + d)^{-r} \lim_{z\to z_0} \frac{A_k Vz - A_k Vz_0}{z - z_0}\,(z - z_0)\, F(z) \\ &= \varepsilon(V)\,(cz_0 + d)^{-r} \left(\frac{d}{dz} A_k Vz\right)_{z=z_0} \cdot \operatorname*{Res}_{z=z_0} F(z).\end{aligned}$$

But

$$\frac{d}{dz}(A_k Vz) = \left(\frac{d}{dVz} A_k Vz\right) \frac{dVz}{dz} = (Vz - p_k)^{-2} (cz + d)^{-2}.$$

Hence,

$$\operatorname*{Res}_{w=A_kVz_0} F(A_k^{-1} w) = \varepsilon(V)\,(cz_0 + d)^{-r-2} (Vz_0 - p_k)^{-2}. \tag{4.3}$$

However, as we saw in (4.1), the summation must be extended only over those poles which lie in the region G_k. This involves a restriction on both the real and imaginary parts of the pole. However, the imaginary part fulfills the required condition automatically. For if $A_k^{-1} w$ is a pole of F, then $e(w/\lambda_k)$ is a pole of f_k, so that $0 < |e(w/\lambda_k)| < \tau_k$, or $\Im w < (-2\pi)^{-1} \lambda_k \log \tau_k$. The required condition is therefore satisfied. On the other hand, with regard to the real part, we have the restriction

$$0 \le \Re A_k Vz_0 < \lambda_k. \tag{4.4}$$

Since $A_k S_k^m Vz_0 = U^{-m\lambda_k} A_k Vz = A_k Vz_0 - m\lambda_k$, we must confine V to a set of representatives of the right cosets of Γ_k in Γ such that each representative satisfies (4.4). Let $\Delta_k(z_0)$ denote such a system:

$$\Delta_k(z_0) = \left\{ V \in \Gamma \,\middle|\, \Gamma = \sum_{V\in\Delta_k} \Gamma_k V,\ 0 \le \Re A_k Vz_0 < \lambda_k \right\}. \tag{4.5}$$

From (4.1) to (4.3) we then obtain

$$(4.6)\quad P_2(z_0) = 2\pi i \sum_{V \in \Delta_k(z_0)} \varepsilon(V)\, e(-(m + \alpha_k)A_k V z_0/\lambda_k)\,(A_k V z_0)^{r+2}(cz_0 + d)^{-r-2},$$

where we have used the relation $A_k Vz = (Vz - p_k)^{-1}$.

This series converges absolutely. To prove this, remember that both the real and imaginary parts of $A_k Vz_0$ are less in absolute value than constants independent of m and N. Hence $|A_k Vz_0| \leq C$. Also,

$$|e(-(m + \alpha_k)A_k Vz_0/\lambda_k)| = \exp\{2\pi(m + \alpha_k)\Im A_k Vz_0/\lambda_k\} \leq \exp Cm,$$

and we see that the series (4.6) is majorized by

$$\exp(Cm) \cdot \sum |cz_0 + d|^{-r-2}.$$

By a result of Poincaré, this series certainly converges, since $r > 0$ (see [4, pp. 201-206] and the remark in [2, top of p. 191]).

If F(z) has several simple poles at $z_1, \cdots, z_q$ with residues $B_1, \cdots, B_q$, the term analogous to P_2 is obviously $\Sigma_{j=1}^q B_j P_2(z_j)$. Now from (3.1), (3.6), and (4.6), we get Theorem 1.

If $r = 0$, the error term CN^{-r} of Section 3 no longer tends to 0 as N increases, and in general, the series (4.6) does not converge. However, in the preceding developments, we can choose $N = \beta\sqrt{m}$ (β = constant), as we did in [2]. The error term E(m, N) (see [2, line following (5.3)]) then becomes O(1). The series (4.6) is summed over the set of V for which $V \in \Delta_k(z_0)$ and $\Im A_k Vz_0 > 1/\beta^2 m$. Theorem 2 follows from (3.1).

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