# TRANSFORMATION GROUPS ON A $K(\pi, 1)$, I

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## 1. INTRODUCTION

The purpose of this note is to give some results on transformation groups and fiberings for a finite-dimensional Eilenberg-MacLane space $K(\pi, 1)$, a space whose one-dimensional homotopy group is $\pi$ and whose remaining homotopy groups vanish. The Eilenberg-MacLane spaces are discussed in [1], and are treated completely in [2]. We are interested here only in the most elementary facts about $K(\pi, 1)$ and $K(\pi, 2)$. Eilenberg and Ganea [3] have pointed out that the existence of a finite-dimensional $K(\pi, 1)$ for a given group $\pi$ is an intrinsic algebraic property of $\pi$.

We were led to this topic by several considerations. For one thing, a special case of transformation groups on a finite-dimensional $K(\pi, 1)$ is quite classical; namely, the study of transformation groups on closed Riemann surfaces of positive genus. Theorem 5.1 contains a generalization of H. A. Schwarz' theorem that no closed Riemann surface of genus larger than 1 can admit a 1-parameter family of complex analytic transformations. The theorem of Montgomery and Samelson [5] to the effect that the only compact connected Lie group which is transitive and effective on a torus is a toral group led us to conjecture and prove that the assumption of transitivity could be dropped. Paul Smith proved that the fixed point set of a cyclic transformation group of prime order p acting on a sphere has the mod p homology groups of a lower-dimensional sphere. In Theorem 3.4 we show that the fixed point set of a cyclic transformation of prime order on a $K(\pi, 1)$ also inherits the mod p homology characteristics of the $K(\pi, 1)$. We feel that Smith's theorem and our Theorem 3.4 are but the two extreme cases of some general relation between the homotopy groups of a space and the cyclic transformations of prime order on that space. This is the real motive for the present note, namely, to initiate the development of extensive relations between homotopy groups and cyclic transformations.

We show that if a finite-dimensional $K = K(\pi, 1)$ is fibered by a connected fiber F with base B, then F is a $K(\pi_1(F), 1)$ and B is a $K(\pi_1(B), 1)$. Our principal result concerns those compact manifolds that are aspherical; that is, $K(\pi, 1)$-spaces that are compact manifolds. We prove that if a compact connected Lie group acts effectively on such a manifold, then the group is a toral group; moreover, this group must act freely, and if $\pi$ is abelian, there is a cross section in the large. Hence, the action might be called a product action.

The $K(\pi, 1)$-spaces are assumed to be connected, locally compact, finite-dimensional ANR's. In this note we use the Alexander-Wallace-Spanier cohomology (AWS-cohomology). We denote a discrete group by $\pi$, a topological group operating on a space by $(G, X)$, and the natural projection of X onto the orbit space X/G by

$$p: X \to X/G.$$

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If $g \in G$, then by $F(g) \subset X$ we denote the set of fixed points under g. Let $F(G) \subset X$ be the set of points fixed under the entire group. A group is said to *operate freely* if F(g) is empty for all g different from the identity. The action is said to be *proper* if, for every $x \in X$, there is an open neighborhood V of x with the property $gV \cap V = \emptyset$ unless g is the identity. Let Y denote the universal covering space of X. If $\pi = \pi_1(X)$ is the fundamental group, then $\pi$ operates naturally on Y as a proper transformation group. Let

$$m: Y \to X$$

denote the covering map.

## 2. A THEOREM ON FIBERINGS

THEOREM 2.1. *Let* [K, B, F, p] *denote a locally trivial fibering of a finite-dimensional space* $K = K(\pi, 1)$, *by a connected fiber* F, *over a base* B. *Then* F *is a* $K(\pi_1(F), 1)$ *and* B *is a* $K(\pi_1(B), 1)$, *and* $\pi_1(F)$ *is injected isomorphically on a normal subgroup of* $\pi$.

Let Y be the universal covering space of $K = K(\pi, 1)$, and let m be the covering map

$$m: Y \to K.$$

The map

$$pm: Y \to B$$

(first m, then p) can be seen to give a fibering of Y over B. A component $F^*$ of one of these fibers is that regular covering space of F which corresponds to the kernel of the injection homomorphism

$$i_*: \pi_1(F) \to \pi_1(K(\pi, 1)).$$

Let X be the space obtained by considering each component of the above fibers as a point, and let b be the map

$$b: Y \to X.$$

Then b is a fiber map with fiber $F^*$. The space X is simply connected, and it is the universal covering space of B. The homotopy sequence of this fibering is then

$$\to \pi_{i+1}(Y) \to \pi_{i+1}(X) \to \pi_i(F^*) \to \pi_i(Y) \to \pi_i(X) \to \cdots.$$

It follows that

$$\pi_{i+1}(X) = \pi_i(F^*) \qquad (i \geq 1),$$

and consequently $\pi_1(F^*)$ is abelian. Hence $\pi_1(F^*)$ is $H_1(F^*; Z)$, the singular one-dimensional homology group over Z.

Now $H_i(F^*; Z) = 0$ $(i > 1)$, as follows for example from Serre [7, p. 467]. Hence $\pi_1(F^*) = 0$. But then $\pi_i(F^*) = H_i(F^*; Z) = 0$ $(i \geq 1)$. This proves that F and B are

$K(\pi, 1)$-spaces. We see that $F^*$ is the universal covering space of F, and that $i_*$ is an isomorphism onto a normal subgroup of $\pi$.

The only compact connected Lie group which is a $K(\pi, 1)$-space is the torus; from this we see at once that if the fibering of the theorem is a principal fibering by a compact connected Lie group, then the group is a torus.

## 3. COVERING TRANSFORMATION GROUPS

We shall assume that X is locally compact, separable metric, arcwise connected, arcwise locally connected and semi-locally 1-connected. We shall assume that (G, X) is a topological transformation group with at least one fixed point. We shall cover (G, X) by a transformation group (G, Y) on the universal covering space of X. In what follows, we select a fixed base point $q \in X$, and it is necessary to select q to be a point fixed under all of G.

Let P(q) be the space of all paths in X beginning at q, topologized as usual by the compact-open topology. Since q is a stationary point, the group G operates on P(q) in a natural way. An equivalence relation is introduced into P(q) by saying that two paths $\alpha(t)$ and $\beta(t)$ are equivalent if $\alpha(1) = \beta(1)$ and if the loop

$$\gamma(t) = \begin{cases} \alpha(2t) & (0 \leq t \leq 1/2), \\ \beta(2(1 - t)) & (1/2 \leq t \leq 1) \end{cases}$$

represents the identity element in $\pi_1(X, q)$. The action of the group G on P(q) preserves this equivalence relation, for $g\gamma(t)$ also represents the identity of $\pi_1(X, q)$, and certainly $g\alpha(1) = g\beta(1)$. The space of equivalence classes in P(q) may be identified with Y; the covering map is obtained by projecting a representative path onto its endpoint.

The action of $\pi_1(X, q) \simeq \pi$ on Y is defined as follows. Let $\sigma(t)$ be a loop in X at q representing $\sigma \in \pi_1(X, q)$. For any path $\alpha(t)$ in P(q), we define a new path

$$(\sigma \circ \alpha)(t) = \begin{cases} \sigma(2t) & (0 \leq t \leq 1/2) \\ \alpha(2t - 1) & (1/2 \leq t \leq 1). \end{cases}$$

The equivalence class of $(\sigma \circ \alpha)(t)$ depends only on $\sigma \in \pi_1(X, q)$ and on the equivalence class of $\alpha(t)$. If $g \in G$, then

$$g(\sigma \circ \alpha)(t) = (g(\sigma) \circ (g\alpha))(t);$$

thus

$$g(\sigma(y)) = (g_*(\sigma))(gy) \tag{1}$$

for $y \in Y$, $g \in G$ and $\sigma \in \pi_1(X, q)$. We see that (G, Y) covers (G, X) in the sense that $m(gy) = gm(y)$. Formula (1) gives us the relation between the actions of G and $\pi = \pi_1(X, x)$ on Y. We shall say that (G, X) is *1-trivial at* q if and only if $g_*(\sigma) = \sigma$ for all $g \in G$ and $\sigma \in \pi_1(X, q)$. If G is 1-trivial at q, then the actions of G and $\pi$ on Y commute. It should be noted that (G, X) is 1-trivial at q if G is arcwise connected. This remark needs only a little clarification. If (G, X) is arcwise connected, then G is 1-trivial at each fixed point of G. Let $q \in X$ be a fixed point, and select

$g \in G$. Let $\alpha(t)$ be a path in G joining e to $g^{-1}$. We define a homotopy $g_t = \alpha(t)g$. Obviously $g_0 = g$ and $g_1 = e$. During the homotopy, q is not moved; hence $g^*: \pi_1(X, q) \to \pi_1(X, q)$ is trivial. Also, we note that if G is 1-trivial at the fixed point q, then it is 1-trivial at every other fixed point which can be joined to q by an arc of fixed points.

LEMMA 3.1. *If* $K(\pi, 1)$ *is finite-dimensional, then* $\pi = \pi_1(K)$ *contains no elements of finite order.*

The universal covering space of K is a contractible finite-dimensional locally compact ANR. Hence the Lemma is a consequence of Smith's fixed-point theorem for periodic maps [8, p. 367].

LEMMA 3.2. *Let* X *be a finite-dimensional* $K(\pi, 1)$*-space, and let* (G, X) *be a* 1*-trivial finite transformation group at the fixed point* q. *For any* $g \in G$, *let* $\tilde{F}(g) \subset Y$ *be the counter-image of* $F(g) \subset X$ *under the covering map. Then* $\tilde{F}(g)$ *is exactly the fixed point set of* g *in the covering action* (G, Y).

Suppose that $m(y) \in F(g)$; then $gy = \sigma y$ for some $\sigma \in \pi_1(X, q)$. Since g has finite order, say r,

$$g^{r-1}gy = g^{r-1}\sigma y = \sigma g^{r-1}y = \sigma^r y = y\,.$$

Now $\pi$ acting on Y has no fixed points and no elements of finite order, so that $\sigma = e$ and $gy = y$.

LEMMA 3.3. *Let* X *be a finite-dimensional* $K(\pi, 1)$*-space. Let* (G, X) *denote a compact connected Lie group acting on* X *with at least one fixed point. Then* $\tilde{F}(G)$, *the counter-image of* F(G) *under the covering map, is the fixed point set of the covering action* (G, Y).

This follows from Lemma 3.2 and the fact that the elements of finite order in a compact connected Lie group are dense. The set $\tilde{F}(G)$ is the intersection of the sets $\tilde{F}(g)$ taken over all elements of finite order in G. This proves the lemma, which will be useful later.

THEOREM 3.4. *Let* K *be a finite-dimensional* $K(\pi, 1)$*-space, and let* $Z_p$ *be the cyclic group of prime order. If* $(Z_p, K)$ *is a* 1*-trivial action at the fixed point* q, *then*

$$i^*: H^*(F(Z_p); Z_p) \simeq H^*(K; Z_p)\,.$$

The proof involves the cohomology of a discrete group (see [1, Chap. 1-13] for a discussion of this concept). It should be noted that $H^*(\pi; Z_p) \simeq H\ (K; Z_p)$ by definition. Let $(Z_p, Y)$ denote the covering action of $(Z_p, K)$. Since Y is a contractible finite-dimensional ANR, it follows from Smith's theorem that the fixed point set $\tilde{F}(Z_p) \subset Y$ is connected, locally connected and acyclic mod p [8, p. 364]. Since the actions of $Z_p$ and $\pi$ on Y commute, there is a proper transformation group $(\pi, \tilde{F}(Z_p))$. The space $\tilde{F}(Z_p)$ is a locally compact, connected, locally connected, closed subset of a separable metric space. Cartan has shown [1, 11-10] that for AWS-cohomology there is a spectral sequence $\{E_r^{s,t}\}$ with

$$E_2^{s,t} \simeq H^s(\pi; H^t(\tilde{F}(Z_p); Z_p))$$

whose $E_\infty$-term is associated with $H^*(F(Z_p); Z_p)$. However, $\tilde{F}(Z_p)$ is acyclic mod p, so that

$$H^s(\pi; Z_p) \approx H^s(F(Z_p); Z_p).$$

We omit the proof that the inclusion map induces the isomorphism. This completes the argument.

THEOREM 3.5. *If* $(T^r, K)$ *denotes an* r-*dimensional toral group acting with at least one fixed point, then*

$$i^*: H^*(F(T^r); Z) \simeq H^*(K; Z).$$

The same proof applies, since $\tilde{F}(T^r) \subset Y$ is acyclic over the integers; we omit the details.

## 4. DIMENSION OF $F(Z_p)$

If $\pi$ has a finite-dimensional $K(\pi, n)$, then the abelian subgroups of $\pi$ can not have arbitrarily large ranks. In fact, the rank can not exceed the dimension of $K(\pi, 1)$. For any $\pi$, let $N(\pi)$ denote the largest possible rank of an abelian subgroup of $\pi$.

THEOREM 4.1. *Let* $(Z_p, K(\pi, 1))$ *denote the action of a cyclic group of prime order on a finite-dimensional* $K(\pi, 1)$ *which is* 1-*trivial at some fixed point* q; *then*

$$1 \leq N(\pi) \leq \dim F(Z_p).$$

This is a lower bound on the dimension of $F(Z_p)$. Let $\tilde{F}(Z_p) \subset Y$ be the fixed point set of $Z_p$ in the covering action $(Z_p, Y)$. Let $H \subset \pi$ be a free abelian subgroup of rank $N(\pi)$; then the action of $\pi$ on Y induces a proper action of H on $\tilde{F}(Z_p)$, which is acyclic mod p; therefore $\tilde{F}(p)/H$ has the cohomology mod p of an $N(\pi)$-dimensional torus, so that $\dim \tilde{F}(Z_p)/H = \dim F(Z_p) \geq N(\pi)$.

The point of Theorem 4.1 is that $N(\pi)$ might be greater than the largest integer for which $H^i(K(\pi, 1); Z_p) \neq 0$. This is the situation for knot groups. If $K \subset S^3$ is the complement of a tame simple closed curve, then K is a 3-dimensional $K(\pi, 1)$. We see that $H^i(K; Z_p) = 0$ $(i > 1)$; however, Whitehead has shown that if the curve is knotted, then $N(\pi) = 2$.

## 5. TRANSFORMATION GROUPS ON COMPACT ASPHERICAL MANIFOLDS

We are ready to state results concerning the action of a compact connected Lie group on a compact connected manifold which is aspherical; that is, on a manifold $M^n$ for which $\pi_i(M^n) = 0$ $(i \geq 2)$.

THEOREM 5.1. *If* $(G, M^n)$ *denotes a compact Lie group acting effectively on a compact, connected, aspherical manifold* $M^n$ *for which* $\chi(M^n) \neq 0$, *then* G *is finite.*

Suppose G contains a circle group $T^1$. Since $\chi(M^n) \neq 0$, the circle group has a nonvoid fixed point set $F(T^1)$ [3]. By Theorem 3.5, $H^n(F(T^1); Z) \simeq H^n(M^n; Z)$; but this implies that $F(T^1) = M^n$. This theorem generalizes the well-known result for compact Riemann surfaces.

THEOREM 5.2. *If* $(G, M^n)$ *denotes a compact connected Lie group acting effectively on a compact, connected, orientable, aspherical manifold, then* G *is a toral group acting freely.*

In view of Theorem 3.4 and the parenthetic remark preceding Lemma 3.1, no element of finite order in G has a fixed point, and hence G acts freely. Gleason [5] has shown that $(G, M^n)$ is a principal fibering of $M^n$ in this case; hence, by Theorem 2.1, G is a toral group. This completes the proof.

As mentioned earlier, it has been shown that the only compact connected Lie group which can operate transitively and effectively on a torus is a toral group [6]. Our Theorem 5.2 is one proof of the fact that the only compact aspherical manifolds which appear as homogeneous spaces of compact connected Lie groups are the tori. If $(G, M^n)$ is transitive on the aspherical manifold $M^n$, we can certainly assume that G is effective. However, G is a toral group acting freely and transitively on $M^n$, so that $M^n$ is homeomorphic to G. Obviously, several other proofs of this remark are possible.

THEOREM 5.3. *Let* $(T^r, M^n)$ *denote a toral group acting effectively on a compact, connected, aspherical manifold* $M^n$ *with abelian fundamental group; then there exists a cross-section of all orbits, and* $M^n$ *is homeomorphic to* $M^n/T^r \times T^r$.

We have just seen that $(T^r, M^n)$ defines a principal fibering of $M^n$ over $M^n/T^r$. It goes almost without saying that $M^n$ has the homotopy type of an n-dimensional torus. By Theorem 2.1,

$$i_*: H_1(T^r; Z) \to H_1(M^n; Z)$$

is a monomorphism. This implies (since $M^n$ and $T^r$ are torsion-free) that

$$i^*: H^i(M^n; Z) \to H^i(T^r; Z)$$

is an epimorphism for $i \geq 0$. The Leray-Hirsch theorem [7] can be applied, since this is a principal bundle, and we conclude that $p^*: H^i(M^n/T^r; Z) \to H^i(M^n; Z)$ is a monomorphism. The principal bundle $[M^n, M^n/T^r, T^r, p]$ is induced by a map [9, p. 101]

$$f: M^n/T^r \to K(Z^r; 2),$$

where $Z^r$ is the r-fold direct sum of the integers with itself. It happens that $K(Z^r; 2)$ is the classifying space of the toral group $T^r$. The image of

$$f^*: H^2(K(Z^r; 2); Z) \to H^2(M^n/T^r; Z)$$

is contained in the kernel of $p^*: H^i(M^n/T^r; Z) \to H^i(M^n; Z)$, from which it follows that $f^*$ is trivial for $i \geq 0$. This can only occur, of course, if f is inessential. It now follows that the bundle $[M^n, M^n/T^r, T^r, p]$ induced over $M^n/T^r$ by f must be a product bundle.

## 6. EXAMPLES

In this section we shall give a few simple examples illustrating some of the results and remarks in this note. We shall describe a process for constructing the type of fiber spaces discussed in Theorem 2.1.

*Construction.* Let $K(\pi_1, 1)$ and $K(\pi_2, 1)$ be two finite-dimensional Eilenberg-MacLane spaces. Let G be any group which operates as a proper transformation group on both $K(\pi_1, 1)$ and $K(\pi_2, 1)$. We form the diagonal transformation group

$$(G, K(\pi_1, 1)\times K(\pi_2, 1))$$

by setting $g(y_1, y_2) = (gy_1, gy_2)$ for $g \in G$. The projection maps of $K(\pi_1, 1)\times K(\pi_2, 1)$ onto the factors gives a diagram

$$\begin{array}{ccc} & (K(\pi_1, 1)\times K(\pi_2, 1))/G & \\ \swarrow \alpha & & \searrow \beta \\ K(\pi_1, 1)/G & & K(\pi_2, 1)/G \end{array}$$

where $\alpha$ and $\beta$ are fiber maps with fiber $K(\pi_2, 1)$ and $K(\pi_1, 1)$, respectively.

The first large class of groups for which $K(\pi, 1)$ can be chosen finite-dimensional is, of course, the class of finitely generated free groups. Next is the class of knot groups; that is, of the fundamental group of the complement of a tame simple closed curve in the 3-sphere. In this case, $K(\pi, 1)$ can be taken to be the complement of the knot. The simplest class of aspherical manifolds are the closed Riemann surfaces of positive genus. Our Theorem 5.1 is, as mentioned earlier, a topological extension of the classical result that a Riemann surface of genus greater than 1 has no nontrivial 1-parameter group of analytic transformations.

Let $M^2$ be a sphere with two handles. The group $Z_2$ acts freely on $M^2$, and since $\chi(M^2/Z_2) = -1$, it follows that the involution reverses the orientation. If we apply the construction to $M^2$, we obtain an aspherical 4-manifold $M^4$ fibered by $M^2$ over $M^2/Z_2$. The involution $(Z_2, M^2\times M^2)$ preserves orientation; therefore $M^4$ is an orientable manifold, and the fibering is nontrivial. A second curious fibering is obtained as follows. Let $\pi$ be the fundamental group of two circles with a point in common. Let Y be the universal covering space. Then $\pi/[\pi, \pi] \simeq Z + Z$ operates freely on the 1-dimensional space $Y/[\pi, \pi]$. The group $Z + Z$ operates as a proper group on the plane $E^2$. If we apply the construction, we obtain a nontrivial fiber bundle over the torus, with fiber $Y/[\pi, \pi]$, whose total space is a 3-dimensional $K(\pi, 1)$.

Finally we should point out the necessity of the hypothesis in Theorem 5.3 that the fundamental group of the total space be abelian. If $\theta \in H^2(T^2; Z)$ is the fundamental class of $T^2$, then the map of $T^2$ into $K(Z, 2)$ representing $\theta$ induces a nontrivial principal bundle $[B^3, T^2, S^1, p]$ over $T^2$. Since $H^1(B^3; Z) \simeq Z + Z$, $\pi_1(B^3)$ is non-abelian, and therefore $B^3$ is not homeomorphic to $T^3$.

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