A NOTE ON BOUNDED CONTINUOUS MATRIX PRODUCTS

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It is well known that the powers A^n of a complex matrix A are uniformly bounded for every positive integer n, if the eigenvalues of A have moduli less than or equal to unity and if the eigenvectors of A include a basis for the vector space. It is not generally true, however, that the products $\Pi_{i=1}^{I} A_i$ associated with an arbitrary sequence of matrices $\{A_i\}$ remain bounded even when each of the matrices has the properties mentioned. A simple counterexample is given by the matrix of the product

$$\begin{pmatrix} 1/2 & 0 \\ 10 & 1/3 \end{pmatrix}\begin{pmatrix} 1/2 & 10 \\ 0 & 1/3 \end{pmatrix},$$

which is easily shown to possess an eigenvalue greater than unity, although the eigenvalues of both factors are distinct and less than unity. In this note, sufficient conditions are established for the products $\Pi_{i=1}^{I} A(\alpha_i)$ to remain bounded when $\{\alpha_i\}$ is a sequence of values of a parameter α upon which a matrix $A(\alpha)$ depends continuously.

The problem arises in relation to the conditions for numerical stability of the solution of linear differential equations by finite difference methods. Its importance is greatest with partial differential equations, but it may be illustrated simply in the case of an equation of ordinary type. As an example, consider the equation:

$$\frac{d^2y}{dx^2} + \frac{1}{x^2}[5 + 3.66 \sin(\pi \log x)]y = 0.$$

The transformations $y = \sqrt{x}u$ and $\log x = 2\theta/\pi - 1/2$ reduce the equation to the general Mathieu form:

$$\frac{d^2u}{d\theta^2} + (1.925 - 2 \times 0.742 \cos 2\theta)u = 0,$$

which has the two independent solutions $u_1 = u_1(\theta)$ and $u_2 = u_2(\theta)$, both of which are uniformly bounded. Consequently the solutions of the original equation are

$$y = \sqrt{x}u_k\left(\frac{\pi}{4} + \frac{\pi}{2}\log x\right) \qquad (k = 1, 2),$$

and these are seen to tend to zero as x approaches the origin positively. A numerical solution is sought on the interval $0 < x \leq \sqrt{e}$ at $x_n = x_0\rho^n$ $(n = 0, 1, 2, \cdots)$ with $x_0 = \sqrt{e}$ and ρ a constant $(0 < \rho < 1)$. The following divided-difference approximation may be constructed:

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$$\frac{d^2y}{dx^2} + \frac{1}{x^2}[5 + 3.66 \sin(\pi \log x)]\, y$$

$$\approx \frac{[y_{n+2} - (1+\rho)y_{n+1} + \rho y_n]}{x_0^2 \rho^n \rho^{n+1}(1-\rho)(1-\rho^2)/2} + \frac{(y_{n+1}/2)}{(x_0^2\rho^n\rho^{n+1}/2)}[5 + 3.66 \sin(\pi \log(x_0 \rho^n))]\,.$$

Point and slope conditions prescribed at $x = \sqrt{e}$ are assumed to be sufficient to determine x_0 and x_1. For convenience of illustration, take $\rho = e^{-1}$; then

$$y_{n+2} + (-1)^n y_{n+1} + 0.37\, y_n = 0 \qquad (n = 0, 1, 2, \cdots).$$

This can be expressed in matrix form as:

$$\begin{pmatrix} y_{2m+1} \\ y_{2m+2} \end{pmatrix} = B\begin{pmatrix} y_{2m} \\ y_{2m+1} \end{pmatrix}, \quad \begin{pmatrix} y_{2m+2} \\ y_{2m+3} \end{pmatrix} = A\begin{pmatrix} y_{2m+1} \\ y_{2m+2} \end{pmatrix} \qquad (m = 0, 1, 2, \cdots),$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -0.37 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -0.37 & -1 \end{pmatrix}.$$

The eigenvalues λ of A and B satisfy $\lambda^2 \mp \lambda + 0.37 = 0$ and $|\lambda| = 0.37$, respectively. Now

$$\begin{pmatrix} y_{2m+2} \\ y_{2m+3} \end{pmatrix} = (A\,B)^m \begin{pmatrix} y_0 \\ y_1 \end{pmatrix},$$

where

$$(A\,B) = \begin{pmatrix} 0 & 1 \\ -0.37 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -0.37 & -1 \end{pmatrix} = \begin{pmatrix} -0.37 & -1 \\ -0.37 & -1.37 \end{pmatrix},$$

which has eigenvalues -0.085 and -1.655. The difference scheme is therefore unstable like $(-1.655)^m$. The example illustrates the manner in which numerical instability may be generated whenever the approximation varies from point to point of the mesh system. This is usually the case if the coefficients of the differential equation are not constant.

In the following theorem, conditions are presented which are sufficient to determine the boundedness of a continued matrix product. Let $\|x\|$ be a norm for the vector space V ($x \in V$); also, let $\|A\| = \sup_{\|x\|=1} \|Ax\|$ be the norm in a complex matrix space M. Let α be contained in a B-space S with norm denoted by $\|\alpha\|_S$, and let $A \equiv A(\alpha)$ be a matrix in M depending continuously on α. The matrix product $\Pi_{i=1}^I A(\alpha_i)$ will be said to be *bounded* in case there is a positive constant C such that

$$\left\| \left(\prod_{i=1}^{I} A(\alpha_i) \right) x \right\| \le C \|x\| \qquad (I = 1, 2, 3, \cdots).$$

THEOREM. *Let* R *be a closed, bounded region in* S, *and let* $\alpha_i \in R$ $(i = 1, 2, 3, \cdots)$. *If there exists a constant* ε $(0 < \varepsilon < 1)$ *such that the eigenvalues* $\lambda(\alpha_i)$ *of* $A(\alpha_i)$ *satisfy the condition* $|\lambda(\alpha_i)| \leq \varepsilon$, *if the quantity* $\Delta = \max \|\alpha_{i+1} - \alpha_i\|_S$ *is less than a certain number* Δ_ε, *and if the eigenvectors of* A *include a basis for* V, *then the matrix product* $\Pi A(\alpha_i)$ *is bounded.*

Proof. Since the eigenvectors of A include a basis for V and the unit eigenvectors of A depend continuously on α, there exists a matrix $E \equiv E(\alpha)$ depending continuously on α, such that $E^{-1}AE = D$, where $D \equiv D(\alpha)$ is the diagonal matrix of eigenvalues of A. Since R is bounded and closed, E depends uniformly on α, and consequently E^{-1} has similar properties. Therefore $\|E^{-1}\|$ takes on a finite maximum $\|E^{-1}\|_{max}$ in R. Let $dE_{i+1} = E_{i+1} - E_i$, where $E_i = E(\alpha_i)$; then the uniform continuity of E implies the existence of $\|dE\|_{max}$ such that, for all i,

$$\|dE_{i+1}\| \leq \|dE\|_{max} \quad \text{and} \quad \lim_{\Delta_\varepsilon \to 0} \|dE\|_{max} = 0.$$

Now

$$E_i = E_{i+1} - dE_{i+1} = (I - dE_{i+1} \cdot E_{i+1}^{-1})E_{i+1},$$

and

$$E_i^{-1} = E_{i+1}^{-1}(I - dE_{i+1} \cdot E_{i+1}^{-1})^{-1} = E_{i+1}^{-1}(I + dE_{i+1} \cdot E_i^{-1}).$$

Define X_i by the relation

$$E_i^{-1} A_i A_{i-1} \cdots A_1 E_1 = D_i D_{i-1} \cdots D_1 + X_i.$$

Then

$$\begin{aligned}(E_{i+1}^{-1} A_{i+1} E_{i+1})(E_i^{-1} A_i A_{i-1} \cdots A_1 E_1) &\\ = E_{i+1}^{-1} A_{i+1} E_{i+1} E_{i+1}^{-1}(I + dE_{i+1} \cdot E_i^{-1})(A_i A_{i-1} \cdots A_1)E_1 &\\ = D_{i+1} D_i \cdots D_1 + D_{i+1} X_i, &\end{aligned}$$

so that

$$\begin{aligned}E_{i+1}^{-1} A_{i+1} A_i \cdots A_1 E_1 &= D_{i+1} D_i \cdots D_1 + D_{i+1} X_i - E_{i+1}^{-1} A_{i+1}\, dE_{i+1} E_i^{-1} A_i A_{i-1} \cdots A_1 E_1 \\ &= D_{i+1} D_i \cdots D_1 + D_{i+1} X_i - D_{i+1} E_{i+1}^{-1}\, dE_{i+1}(D_i D_{i-1} \cdots D_1 + X_i).\end{aligned}$$

Thus

$$X_{i+1} = D_{i+1}(I - E_{i+1}^{-1} \cdot dE_{i+1})X_i - D_{i+1} E_{i+1}^{-1}\, dE_{i+1}(D_i D_{i-1} \cdots D_1).$$

If we write $\|dE\|_{max} \|E^{-1}\|_{max} = \delta$, it follows that

$$\|X_{i+1}\| \leq (1 + \delta) \|D_{i+1}\| \|X_i\| + \delta \prod_{k=1}^{i+1} \|D_k\|.$$

Since $\|D_k\| \leq \varepsilon$, this recursion relationship is dominated by

$$\|X_{i+1}\| = \varepsilon(1 + \delta)\,\|X_i\| + \varepsilon^{i+1}\,\delta\,.$$

Since $X_1 = 0$, this in turn has the solution $\|X_i\| = \varepsilon^i[(1 + \delta)^{i-1} - 1]$; and this is bounded if $\delta \leq 1/\varepsilon - 1$, that is, if Δ_ε is sufficiently small.

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