DOUBLY STOCHASTIC MEASURES

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1. INTRODUCTION

A *doubly stochastic* $n \times n$ *matrix* has nonnegative elements, and the sum of each row and each column is one. A *permutation matrix* is a doubly stochastic matrix in which the elements are zero or one. Thus the identity matrix is a permutation matrix. Many authors, including J. von Neumann [3] and G. Birkhoff [1], have shown that the set of doubly stochastic matrices is the convex hull of the permutation matrices.

We may also consider matrices with a countable infinity of rows and columns. Let $\mathscr{M}_0$ denote the space of all infinite matrices M whose row- and column-sums are absolutely convergent. We say that the net $\{M_\alpha\}$ in $\mathscr{M}_0$ converges to zero if, for each row and each column, the absolute sum converges to zero with α. With this topology, Rattray and Peck [4] have shown that the closure in $\mathscr{M}_0$ of the convex hull of the permutation matrices is the set of all doubly stochastic matrices.

2. DOUBLY STOCHASTIC MEASURES

Let X be the half-line $[0, \infty)$, and consider the quadrant $X \times X$. (We use the half-line for simplicity of notation; the results which follow hold also when X is the whole real line.) We shall denote by $\mathscr{M}$ the set of all real (not necessarily nonnegative) measures μ, defined on the measurable subsets of $X \times X$, which are such that, for each measurable set $E \subset X$ of finite Lebesgue measure $\lambda(E)$, the total variation [2, Section 39] of μ on $E \times X$ and on $X \times E$ is finite. In what follows, λ will always indicate Lebesgue measure.

If μ is a nonnegative measure on $X \times X$ such that $\mu(E \times X) = \mu(X \times E) = \lambda(E)$ for all measurable $E \subset X$, then μ will be called a *doubly stochastic measure* and $\mu \in \mathscr{M}$. A set S is a support of a nonnegative measure μ if $\mu(X \times X \sim S) = 0$. If S is a support of a doubly stochastic measure μ and is such that every section $\{y: (x, y) \in S\}$ $(x \in X)$ and $\{x: (x, y) \in S\}$ $(y \in Y)$ consists of one point only, then μ will be called a *permutation measure*. Thus the measure δ which has the diagonal $\Delta = \{(x, x): x \in X\}$ as a support and is defined by $\delta(E) = \lambda\{x: (x, x) \in E\}$ for $E \subset \Delta$ is the simplest permutation measure.

Not all doubly stochastic measures are like this. If f is a nonnegative measurable function on $X \times X$ such that $\int_X f(x, y)\, d\lambda(x) = 1$ for all y, and $\int_X f(x, y) d\lambda(y) = 1$ for all x, then f may be called a *doubly stochastic function*. For any such f, the measure μ defined by $\mu(E) = \int_E f(x, y)\, d\lambda(x, y)$ for $E \subset X \times X$ will be a doubly stochastic measure. However, δ can not be so expressed. If ψ is a nonnegative, even, measurable function on the real line such that $\int_{-\infty}^{\infty} \psi(t)\, d\lambda(t) = 1$, then $\psi(x - y)$

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is doubly stochastic on the plane and $\psi(x - y) + \psi(x + y)$ is doubly stochastic in a quadrant. Thus there is a large class of doubly stochastic measures.

A measure σ, defined on the square $A = \{(x, y): 0 \leq x < a, 0 \leq y < a\}$ by $\sigma(E) = r\delta(E \cap A)$ for some real r and all measurable $E \subset X \times X$, will be called a *spine* on A. Thus $\sigma(X \times X) = \sigma(A) = ra$. Similarly, we may define a spine on any square, say $B = \{(x, y): b_1 \leq x < b_1 + a, b_2 \leq y < b_2 + a\}$, by translation. Thus if $\sigma'(E + (b_1, b_2)) = \sigma(E)$, with $\overline{B}$ a support of σ', then σ' is a spine on B, with $\sigma'(X \times X) = \sigma'(B) = ra$. A doubly infinite array of spines on congruent squares which disjointly cover $X \times X$ will be called a *spine matrix*. It is close to the concept of an infinite matrix, and we shall exploit this similarity.

3. THE TOPOLOGY OF $\mathcal{M}$

The set of doubly stochastic measures is a convex subset of $\mathcal{M}$, and each permutation measure is an extreme point. We set up a topology in $\mathcal{M}$ by considering the set Φ of all bounded uniformly continuous functions ϕ on X. A neighbourhood of μ in $\mathcal{M}$ is defined by means of real numbers $a > 0$, $\varepsilon > 0$, and a finite set $\{\phi_1, \cdots, \phi_q\} \subset \Phi$. We shall say that $\nu \in N_{a,\varepsilon,\phi_1,\cdots,\phi_q}(\mu)$ if for all $E = [0, t)$, where $0 < t \leq a$,

$$\left| \int_{X\times E} \phi_p(x)\, d\mu(x, y) - \int_{X\times E} \phi_p(x)\, d\nu(x, y) \right| < \varepsilon,$$

$$\left| \int_{E\times X} \phi_p(y)\, d\mu(x, y) - \int_{E\times X} \phi_p(y)\, d\nu(x, y) \right| < \varepsilon,$$

for $1 \leq p \leq q$. The use of the family Φ is suggested by the weak topology of probability measures on X when considered as linear functionals on Φ, for in this topology the probability measures are closed. Also, the topology here is defined stripwise in each direction in conformity with that used in [4] for infinite doubly stochastic matrices. As an indication of what the topology on $\mathcal{M}$ may accomplish, we note the fact (though we shall not use it) that if χ_n is the characteristic function of the interval $[-1/n, +1/n]$, then the sequence of doubly stochastic measures generated by the doubly stochastic functions $\frac{n}{2}[\chi_n(x + y) + \chi_n(x - y)]$ converges to the permutation measure δ.

Since the constant function 1 is in Φ, it is clear that the convex set of doubly stochastic measures is a closed subset of $\mathcal{M}$. We shall prove that *the closure in $\mathcal{M}$ of the convex hull of the permutation measures is the set of all doubly stochastic measures.*

Consider any $\mu \in \mathcal{M}$ which is doubly stochastic. Let $N_{a,\varepsilon,\phi_1,\cdots,\phi_q}(\mu)$ be any neighbourhood of μ, and let

$$\sup\{ |\phi_p(x)| : x \in X,\ 1 \leq p \leq q\} = K.$$

Since the ϕ_p are uniformly continuous, we may find a b such that $0 < b < \varepsilon/4K$, and such that the oscillation of each ϕ_p on each interval of length b is less than $\varepsilon/4a$.

Let $A_{ij} = \{(x, y): ib \leq x < (i+1)b,\ jb \leq y < (j+1)b\}$ for $i, j = 0, 2, \cdots$; then A_{ij} is a collection of congruent squares which disjointly cover $X \times X$. For each pair i, j, let σ_{ij} be the spine on the square A_{ij} with $\sigma_{ij}(X \times X) = \sigma_{ij}(A_{ij}) = \mu(A_{ij})$, and write $\nu = \Sigma_{i,j}\sigma_{ij}$. The nonnegative measure ν is in fact doubly stochastic by construction, and it is a spine matrix defined on the A_{ij}'s. We shall show that $\nu \in N_{a,\varepsilon,\phi_1,\cdots,\phi_q}(\mu)$. In fact, if $0 < t \leq a$, we shall write

$$B = \{(x, y): 0 \leq y < [t/b]b\}, \qquad C = \{(x, y): [t/b]b \leq y < t\}.$$

Note that $B = \bigcup_{i<\infty, j<t/b} A_{ij}$; that if $E = [0, t)$, we have $X \times E = B \cup C$, and that this union is disjoint.

Since the oscillation of each ϕ_p is less than $\varepsilon/4a$ on every A_{ij}, and since $\nu(A_{ij}) = \mu(A_{ij})$, we have

$$\left|\int_{A_{ij}} \phi_p(x)\, d\mu(x, y) - \int_{A_{ij}} \phi_p(x)\, d\nu(x, y)\right|$$

$$\leq \left|\int_{A_{ij}} \phi_p(x)\, d\mu(x, y) - \phi_p(ib)\, \mu(A_{ij})\right|$$

$$+ \left|\phi_p(ib)\, \nu(A_{ij}) - \int_{A_{ij}} \phi_p(x)\, d\nu(x, y)\right|$$

$$< (\varepsilon/4a)\, \mu(A_{ij}) + (\varepsilon/4a)\, \nu(A_{ij}) = (\varepsilon/2a)\, \mu(A_{ij}).$$

Thus

$$\left|\int_B \phi_p(x)\, d\mu(x, y) - \int_B \phi_p(x)\, d\nu(x, y)\right| < (\varepsilon/2a)\, \mu(B).$$

Also, b is chosen small enough so that

$$\left|\int_C \phi_p(x)\, d\mu(x, y)\right| + \left|\int_C \phi_p(x)\, d\nu(x, y)\right| < 2Kb < \varepsilon/2.$$

Putting these two inequalities together, and observing that $\mu(B) \leq a$, we have for $1 \leq p \leq q$ that

$$\left|\int_{X \times E} \phi_p(x)\, d\mu(x, y) - \int_{X \times E} \phi_p(x)\, d\nu(x, y)\right| < \varepsilon.$$

Similarly,

$$\left|\int_{E \times X} \phi_p(y)\, d\mu(x, y) - \int_{E \times X} \phi_p(y)\, d\nu(x, y)\right| < \varepsilon.$$

Therefore $\nu \in N_{a,\varepsilon,\phi_1,\cdots,\phi_q}(\mu)$.

Finally if $\{\tau_\alpha\}$ is a net of spine matrices on the A_{ij}'s and if, for each j, $A_j = \bigcup_i A_{ij}$, we see that $\lim_\alpha \Sigma_i |\tau_\alpha(A_{ij})| = 0$ implies that

$$\lim_\alpha \left| \int_{A_j} \phi(x)\, d\tau_\alpha(x, y) \right| = 0$$

for each $\phi \in \Phi$. This means that the topology which $\mathscr{M}$ induces on the spine matrices is not finer than that used in [4] for infinite matrices. Therefore by the result in [4], every neighbourhood of ν, and therefore $N_{a,\varepsilon,\phi_1,\cdots,\phi_q}(\mu)$, contains an element of the convex hull of those spine matrices which are permutation measures.

4. MEASURES ON A SQUARE

One may define doubly stochastic and permutation measures in the case where X is a closed interval, say $X = [0, 1]$. A similar theorem holds in this case, and all who have read the above may supply the statement and the proof, using the theorem in [3] or [1].

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