AN EXTENSION THEOREM FOR A CLASS OF DIFFERENTIAL OPERATORS

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1. INTRODUCTION. The principal theorem of this paper arises in the study of the behavior of analytic functions on the boundary of a disk, in the study of smoothing operators, and in higher-order generalizations of the Poincaré-Bendixson gradient theorem. The class of differential operators involved was first studied by Loewner [2] who showed that the curves generated by our operators (3) have the property of nonnegative circulation, that is, have nonnegative order with respect to each point. As is well known, a function of a complex variable which is analytic in a disk and continuous on the closure of the disk maps the boundary of the disk into a curve of nonnegative circulation. Later in this paper, however, we give an example of a curve of nonnegative circulation which is not such an image, even after any change of parametrization that does not change the curve's topological character. In fact, our curve is not the image of the boundary of the disk under any mapping which is light and interior on the interior of the disk, and which is thus topologically equivalent to an analytic function on the interior (Stoilow [3]). Our principal theorem shows, however, that Loewner's curves are such images; thus it proves that they form a proper subclass of the curves of nonnegative circulation. Indeed, by using a result of Jewett [1], we show that for Loewner's curves the mapping on the open disk can also be taken to be n-times differentiable.

In later work we hope to pursue this further, both in the direction of more information about the light interior function, and in the direction of integral operators.

2. Let D denote the closed unit disk in the xy-plane, and let s denote the positively oriented unit circle which bounds D.

Let $P_n(r)$ and $P_{n-1}(r)$ be a pair of polynomials with real coefficients, of degree n and n - 1, respectively:

$$(1)\qquad \begin{cases} P_n(r) = p_n^0 r^n + p_n^1 r^{n-1} + \cdots + p_n^n, \\ P_{n-1}(r) = p_{n-1}^0 r^{n-1} + p_{n-1}^1 r^{n-2} + \cdots + p_{n-1}^{n-1}, \end{cases}$$

such that

(2)

(a) $p_n^0 > 0,\ p_{n-1}^0 > 0;$

(b) the roots of P_n and P_{n-1} are real and simple;

(c) the roots of P_{n-1} separate the roots of P_n; that is, if r_n^i and r_{n-1}^i are the roots of P_n and P_{n-1}, respectively, then $r_n^1 < r_{n-1}^1 < r_n^2 < \cdots < r_{n-1}^{n-1} < r_n^n$.

Let f(t) be a real-valued function in C^{n+1} defined over s, where t is the real-angle parameter $(0 \le t \le 2\pi)$. Consider the pair of differential operators obtained from polynomials in (1) as applied to the function f(t):

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$$(3)\qquad \begin{cases} u = P_n[f(t)] = p_n^0 f^{(n)}(t) + p_n^1 f^{(n-1)}(t) + \cdots p_n^n f(t), \\ v = P_{n-1}[f(t)] = p_{n-1}^0 f^{(n-1)}(t) + p_n^1 f^{(n-2)}(t) + \cdots + p_{n-1}^{n-1} f(t). \end{cases}$$

This pair of operators defines a continuous mapping of the positively oriented circle s in the xy-plane into the uv-plane. The curve in (3) is oriented and, of course, closed. Let us further assume that f(t) is such that the curve δ,

$$(4)\qquad u = f'(t), \quad v = f(t),$$

(5)
- (a) intersects the v-axis a finite number of times; and has the property that
- (b) if δ intersects the v-axis for some t, then δ crosses the v-axis for that t.

The curve in (3), where $P_n(r)$ and $P_{n-1}(r)$ satisfy conditions (2) and where f(t) satisfies conditions (4) and (5), will be called an L-curve [2].

A mapping I will be called an i-mapping of D if

(6)
- (a) I is continuous on D,
- (b) I is light interior on Int D,
- (c) I is sense-preserving on Int D.

If α is a curve and I an i-mapping such that, for each point t on s, $\alpha(t) = I(t)$, then α will be called an i-boundary. The principal theorem of this paper is the following.

THEOREM 1. *Every* L-*curve is an* i-*boundary.*

Jewett has proved that, for each positive integer n, every i-map f can be approximated arbitrarily closely by maps that are continuous on D, are light, interior and of class C^n on Int D, and agree with f on Bdry D.

3. We first prove a theorem, a special case of which will be used as a lemma for the principal theorem. We need a few more definitions.

Let α be a continuous mapping of the positively oriented unit circle s into the uv-plane which satisfies the following conditions:

(7)
- (a) For at most a finite number of points $P_1, P_2, \cdots, P_n$ on s does $v_i = \alpha(P_i)$ lie on the v-axis. We suppose that $n \neq 0$ and that the points P_i are in positive cyclic order on s.
- (b) Each point P_i lies in an interval of s that is mapped homeomorphically onto an arc that crosses the v-axis and intersects it only at v_i.
- (c) For each i, $v_i - v_{i-1}$ and $v_{i+1} - v_i$ are not zero and are of opposite sign, subscripts being read modulo n.
- (d) Each arc $a_i = P_i P_{i+1}$ is mapped homeomorphically by α onto an arc α_i.

Note that n is necessarily an even number; in the following, we write n = 2m.

We now define a special subdivision of D, which will be one of the essential tools. There exists a nonempty class $\triangle$ of subdivisions of D into closed 2-cells D_i such that

(8)

(a) $\bigcup_{i=1}^{2m} D_i = D$;

(b) the intersection $D_i \cap D_j$ of any two of the cells is either an arc or a point, or it is empty; consequently, no interior points of one cell belong to another;

(c) $D_i \cap s = a_i$, which is an arc whose endpoints lie in $\bigcup_{i=1}^{2m} P_i$.

Under these conditions the union of the boundaries of the cells D_i is a topological realization of a 1-complex. There is, however, considerable freedom in the selection of such subdivisions. For example, for $2m = 4$, one such subdivision is that affected naturally by the coordinate axes; another is obtained from the three line segments

C_1: from (1, 0) to (0, 1),

C_2: from (-1, 0) to (0, -1),

C_3: from the mid-point of C_1 to the midpoint of C_2.

We remark that *condition* (b) *of* (8) *follows from* (a) *and* (c). For if $D_i \cap D_j$ is not connected, then by the Mullikan Theorem [4] $D_i \cup D_j$ separates the plane. One complementary domain, E, of $D_i \cup D_j$ lies in D, and therefore it intersects some D_k. But Int D_k is in E, so that (Bdry $D_k) \cap s$ is in $(D_i \cup D_j) \cap s$. If $2m = 2$, our claim is certainly true; if $2m \geq 4$, then $(D_i \cup D_j) \cap s$ consists of at most one point, so that $D_j \cap s$ consists of one point, contradicting (c).

THEOREM 2. *Corresponding to each curve* α *satisfying conditions* (7), *there exists a subdivision in* $\triangle$ *and a mapping* H *such that* H *is a sense-preserving homeomorphism over each* D_i *of the subdivision, continuous on* D, *and such that* H *agrees with* α *on* s *and maps that part of the* 1-*complex of the subdivision which is interior to* D *into the* v-*axis.*

Proof. Under the conditions on a subdivision in $\triangle$, the set $D_i \cap D_{i+1}$ is an arc. For, first, $D_i \cap D_{i+1}$ is not empty. If $D_i \cap D_{i+1}$ is a point P, then $P = a_i \cap a_{i+1}$. Since P is not in the boundary of any other D_j, there exists an open set U containing P but containing no point of any other such set. However, in $U \cap D$ there exists an arc from some point of D_i to some point of D_{i+1} that does not contain P. This arc, then, must intersect $D_i \cap D_{i+1}$, a contradiction.

We proceed to the proof of the theorem by induction:

Part 1: m = 1. Let α_1 denote the closed arc of α in the right half-plane, and α_2 the closed arc in the left half-plane. Let β_1 be the oriented interval from v_2 to v_1 and β_2 the oriented interval from v_1 to v_2. Let γ_k denote the union of α_k and β_k, both of which are then positively oriented Jordan curves. Let Γ_k be the region bounded and oriented by γ_k. Define a subdivision in $\triangle$ as follows:

$$D_1 = \{(x, y) \mid x \geq 0\} \cap D,$$

$$D_2 = \{(x, y) \mid x \leq 0\} \cap D.$$

There exists then a mapping H over $D = D_1 \cup D_2$ which maps D_k into Γ_k homeomorphically, $D_1 \cap D_2$ onto the v-axis, and agrees with α on s.

Part 2: Assume the theorem holds for $m = n - 1$. Let α be a curve satisfying conditions (7) and intersecting the v-axis in $2(m + 1)$ points v_i. Let a_i be one of the $2(m + 1)$ intervals on s which is mapped into α, and such that the image α_i lies in one half-plane and the end points of α_i lie on the v-axis: Bdry $\alpha_i = (v_i, v_{i+1})$.

Let β_i denote the closed oriented interval on the v-axis from v_{i+1} to v_i, where $\gamma_i = \alpha_i \cup \beta_i$ is a positively oriented Jordan curve. Let μ be a function which assigns to each β_i an integer in the following way:

$\mu_i = \mu(\beta_i)$ is the number of β_σ which contains β_i $(\beta_\sigma \supset \beta_i)$.

There exists then a μ_k which is a maximum in the sense that

$$\mu_{k-1} \leq \mu_k \geq \mu_{k+1},$$

from which it immediately follows that

$$\beta_{k-1} \supset \beta_k \subset \beta_{k+1}. \tag{9}$$

Consider the part of α composed of $\alpha_{k-1} \cup \alpha_k \cup \alpha_{k+1}$; note that α_{k-1} and α_{k+1} lie in one half-plane and α_k in the other. It follows from (9) that α_{k-1} and α_{k+1} have at least one point in common. Let π be the first point on α_{k-1} which is in $\alpha_{k-1} \cap \alpha_{k+1}$. Consider now the curve $\tilde{\alpha}$ which is defined as follows:

on a_i $(1 \leq i \leq k - 2)$, $\tilde{\alpha} = \alpha$;

on a_{i+2} $(k \leq i \leq 2m)$, $\tilde{\alpha} = \alpha$;

on $a_{k-1} \cup a_k \cup a_{k+1}$, $\tilde{\alpha}$ is a homeomorphism onto the arc which is the union of the arc of α_{k-1} from v_{k-1} to π and the arc of α_{k+1} from π to v_{k+2}.

By the induction hypothesis, there exists for $\tilde{\alpha}$ a subdivision $\tilde{S}$ in Δ and a mapping $\tilde{H}$ of D. Let $\tilde{D}_i$ $(i = 1, 2, \cdots, 2m)$ denote the cells of the subdivision $\tilde{S}$, where the $\tilde{a}_i = \tilde{D}_i \cap s$ are then related to the a_i for α as follows:

$$\tilde{a}_i = a_i \quad (1 \leq i \leq k - 2),$$

$$\tilde{a}_i = a_{i+2} \quad (k \leq i \leq 2m),$$

$$\tilde{a}_i = a_{k-1} \cup a_k \cup a_{k+1}.$$

We shall now define the required subdivision S and the mapping H, with the aid of the subdivision $\tilde{S}$ and the mapping $\tilde{H}$.

Let P be a point in $\tilde{H}^{-1}(\beta_k) \cap D_{k-1} \cap$ Int D. Let (P_{k-1}, P_k), (P_k, P_{k+1}) and (P_{k+1}, P_{k+2}) denote the boundaries of a_{k-1}, a_k and a_{k+1}, respectively. Let d_k denote an arc from P_k to P with its interior points in Int D_{k-1} $(\tilde{D}_{k-1} \cap s = \tilde{a}_{k-1} = a_{k-1} \cup a_k \cup a_{k+1})$, and let d_{k+1} denote an arc from P_{k+1} to P with its interior points in Int $\tilde{D}_{k-1}$ and not intersecting d_k. Let d_{k-1} denote the arc in Bdry $\tilde{D}_{k-1}$ from P_{k-1} to P, and d_{k+2} the arc in Bdry $\tilde{D}_{k-1}$ from P_{k+2} to P. We now define our subdivision S, which will clearly be in Δ, as follows:

$$D_i = \tilde{D}_i \quad (1 \leq i \leq k - 2),$$

$$D_{i+2} = \tilde{D}_i \quad (k \leq i \leq 2m),$$

D_{k-1} is the closed 2-cell bounded by $a_{k-1} \cup d_{k-1} \cup d_k$,

D_k is the closed 2-cell bounded by $a_k \cup d_k \cup d_{k+1}$,

D_{k+1} is the closed 2-cell bounded by $a_{k+1} \cup d_{k+1} \cup d_{k+2}$.

We now define H with the aid of $\tilde{H}$ and our subdivision S in Δ. Let

$$H(D_i) = \tilde{H}(D_i), \quad D_i = \tilde{D}_i \quad (1 \leq i \leq k - 2),$$

$$H(D_{i+2}) = \tilde{H}(D_{i+2}), \quad D_{i+2} = \tilde{D}_i \quad (k \leq i \leq 2m).$$

To define H over D_{k-1}, D_k and D_{k+1}, we first let Γ_i be the region in the uv-plane bounded by the positively oriented Jordan curve $\gamma_i = \alpha_i \cup \beta_i$. We note, for H over D_{k-1}, that H is already defined on a_i by α, and on d_{k-1} from H over

$$\left(\bigcup_{i=1}^{k-2} D_i\right) \cup \left(\bigcup_{i=k+2}^{2m+2} D_i\right).$$

This part of H on Bdry D_{k-1} can be extended to a homeomorphism of the whole positively oriented Bdry D_{k-1} onto γ_{k-1}. It follows then that $d_{k-1} \cup d_k$ is mapped into the v-axis. Let $H(D_{k-1})$ be a sense-preserving homeomorphic extension of this mapping to D_{k-1}.

The problem of defining H over D_k and D_{k-1} is essentially the same. Treating H over D_k first, we see that the mapping is already defined over a_k by α and over d_k by $H(D_{k-1})$. We can then extend this mapping to a homeomorphism of the positively oriented Bdry D_k onto γ_k. We now let $H(D_k)$ be a sense-preserving homeomorphic extension of this mapping to D_k. Finally, over D_{k+1} the mapping is already given as a homeomorphism of the positively oriented Bdry D_{k+1} onto γ_{k+1}. Let $H(D_{k+1})$ then be a sense-preserving homeomorphic extension of this mapping to D_{k+1}.

Our mapping H is now defined over the whole of D, and we can easily see that it satisfies the required conditions. The proof of Theorem 2 is thus completed.

THEOREM 3. *Every mapping* H *is an* i-*mapping.*

Proof. We must show that H satisfies the conditions (6). Condition (a) is trivially verified. Condition (c) follows from the fact that H is a sense-preserving homeomorphism over each D_i of its subdivision in Δ. For condition (b) we first prove *lightness:* the inverse image of a point can have at most one point in each of the D_i; hence the mapping is at most 2m-to-one. As to *interiority*, let U be a spherical neighborhood in Int D. If U is contained in some D_k, then the image of U is clearly open, since H is a homeomorphism over D_k. If U intersects exactly two 2-cells, say D_j and D_k, then the image of Int $(U \cap D_j)$ and Int $(U \cap D_k)$ are each open. Call these images v_j and v_k, respectively. The arc $(D_j \cap D_k)$ is mapped into an arc which separates v_j and v_k in the image of U. It follows that U is open. There exists at most a finite number of points at which three or more D_i intersect. The interiority at these points now follows from a theorem of Whyburn [5, p. 150].

The following two results, the first of which is obvious, give conditions that a mapping (4) be an i-boundary. The second theorem is purely analytic.

THEOREM 4. *Let* $f(t)$ *be a real-valued function of period* 2π *in* C^1. *Suppose also that* $f'(t)$ *has a finite number of zeros in* $[0, 2\pi]$, *and that* $f'(t)$ *changes sign at each of its zeros. Then the curve* α:

$$u = f'(t), \qquad v = f(t) \qquad (0 \leq t \leq 2\pi),$$

has the property that

v *is strictly increasing if and only if* $u > 0$,

v *is strictly decreasing if and only if* $u < 0$.

The curve α in the theorem satisfies the hypothesis of Theorem 1, and thus, by Theorem 3, it is an i-boundary. The following theorem gives a simple analytical condition on f that implies the hypothesis of Theorem 4.

THEOREM 5. *Let* $f(t)$ *be a real-valued function of period* 2π *in* C^2, *and suppose also that* f' *and* f'' *have no common zero. Then* $f(t)$ *satisfies the assumptions in Theorem 4.*

To prove this statement, note first that if $f'(t_0) = 0$, then $f''(t_0) \neq 0$, and thus $f'(t)$ changes sign as t moves through t_0. Second, if f' had an infinite number of zeros, then f'' would also, and further it would have a zero in common with f'.

4. We now proceed with the proof of Theorem 1. Given $P_n(r)$ and $P_{n-1}(r)$, we can write

$$P_n(r) = Q_n(r)\, P_{n-1}(r) - P_{n-2}(r), \tag{10}$$

where $Q_n(r) = q_n^0 r + q_n^1$, $q_n^0 > 0$, and where $P_{n-1}(r)$ and $P_{n-2}(r)$ satisfy conditions (2) for $n - 1$ (for a proof of this assertion, see [2, Lemma 1]). This algorithm can be continued, giving rise to the sequence of polynomials $P_n(r)$, $P_{n-1}(r)$, $\cdots$, $P_0(r)$, where $P_0(r) = p_0^0$. Corresponding to (10), the differential formula

$$P_n(f) = Q_n[f]\, P_{n-1}[f] - P_{n-2}[f] = q_n^0\, P_{n-1}[f'] + q_n^1\, P_{n-1}[f] - P_{n-2}[f] \tag{11}$$

holds. Hence, there is the sequence of curves

$$u = P_k[f], \qquad v = P_{k-1}[f] \qquad (k = 1, 2, \cdots, n).$$

By Theorem 4, there exists an i-mapping, which we shall denote by I_1, such that $I_1(s) = \begin{pmatrix} f' \\ f \end{pmatrix}$ Let A denote the proper affine mapping which we identify with the matrix

$$A_1 = \begin{pmatrix} p_0^1 & p_1^1 \\ 0 & p_0^0 \end{pmatrix} \qquad (\det A_1 = p_0^1 p_0^0 > 0).$$

The mapping $A_1 I_1$ is also an i-mapping, and

$$A_1 I_1(s) = \begin{pmatrix} P_1[f] \\ P_0[f] \end{pmatrix}.$$

Let A_k denote the proper affine mapping which we identify with the matrix

$$A_k = \begin{pmatrix} q_k^1 & -1 \\ 1 & 0 \end{pmatrix} \quad (k = 2, 3, \cdots, n).$$

Let $E_k = \{(x, y) \mid k - 1 \leq \sqrt{x^2 + y^2} \leq k\} (k = 2, 3, \cdots, n)$. Let I_k denote the mapping of E^k defined by

$$\text{(12)} \qquad \begin{aligned} u &= P_{k-1}[f], \\ v &= -(\rho - k + 1) q_{k-1}^0 P_{k-1}[f'] + P_{k-2}[f], \qquad (0 \leq t \leq 2,\ k - 1 \leq \rho \leq k). \end{aligned}$$

where $x = \rho \cos t$, $y = \rho \sin t$. Note that the polar-coordinate Jacobian of I_{k+1} is

$$\text{(13)} \qquad J(I_{k+1}) = \begin{vmatrix} 0 & P_k[f'] \\ -q_k^0 P_k[f'] & * \end{vmatrix} \geq 0.$$

Over $D^n = \{(x, y) \mid \sqrt{x^2 + y^2} \leq n\}$ we define a mapping I_n^*, where

$$\text{(14)} \qquad I_n^* = \begin{cases} F_1 = A_n A_{n-1} \cdots A_1 I_1 & \text{over } E_1, \\ F_2 = A_n A_{n-1} \cdots A_2 I_2 & \text{over } E_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ F_{n-1} = A_n A_{n-1} I_{n-1} & \text{over } E_{n-1}, \\ F_n = A_n I_n & \text{over } E_n. \end{cases}$$

We first show that I_n^* is continuous over D^n. That the mappings F_j are continuous over E_j follows immediately from their definitions. All we have to show is that F_j and F_{j+1} agree, wherever both are defined, namely on $E_j \cap E_{j+1}$. Let s_k denote the positively oriented unit circle of radius k which lies then on $E_k \cap E_{k+1}$. The mapping $I_n^*(s_k)$ is defined both as $F_k(s_k)$ and as $F_{k+1}(s_k)$. We have

$$F_k(s_k) = A_n A_{n-1} \cdots A_k I_k(s_k);$$

but

$$A_k I_k(s_k) = \begin{pmatrix} q_k' & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_{k-1}[f] \\ -q_{k-1} P_{k-1}[f'] + P_{k-2}[f] \end{pmatrix},$$

and by the differential formula (11) we then have

$$A_k I_k(s_k) = \begin{pmatrix} P_k[f] \\ P_{k-1}[f] \end{pmatrix};$$

thus

$$F_k(s_k) = A_n A_{n-1} \cdots A_{k+1} \begin{pmatrix} P_k[f] \\ P_{k-1}[f] \end{pmatrix} \qquad (k = 1, 2, \cdots, n). \tag{14}$$

On the other hand we have $F_{k+1}(s_k) = A_n A_{n-1} \cdots A_{k+1} I_{k+1}(s_k)$, and since

$$I_{k+1}(s_k) = \begin{pmatrix} P_k[f] \\ P_{k+1}[f] \end{pmatrix},$$

we have $F_{k+1}(s_k) = F_k(s_k)$ $(k = 1, 2, \cdots, n - 1)$. Thus I_n^* is continuous on D^n; it also follows from (14), for $k = n$, that

$$I_n^*(s_n) = \begin{pmatrix} P_n[f] \\ P_{n-1}[f] \end{pmatrix} = \alpha .$$

Note next that the Jacobian of I_n^* is defined and is continuous in each closed ring E_k $(k = 2, 3, \cdots, n)$. Explicitly, the polar-coordinate Jacobian of F_k is

$$J(F_k) = (\det A_n)(\det A_{n-1}) \cdots (\det A_k)\, J(I_k) \geq 0,$$

since

$$J(I_k) \geq 0 \qquad \text{and} \qquad \det A_i = 1 \qquad (i \geq 2).$$

Although the Jacobian of I_n^* is continuous in each E_k $(k \geq 2)$, it need clearly not be continuous on the circle s_k. Further, since we have only topological control over $D_1 = E_1$, the Jacobian may not even be defined in E_1.

Let J_k be the set of points in E_k for which $J(F_k)$ vanishes. If $P \in J_k$, then the radial line through P in E_k is also in J_k. This follows from the fact that the Jacobian (13) is independent of ρ. From the mapping (12) we then notice that each such radial line is mapped into a point.

Let Z be the union of the J_k, and D_n the union of the E_k. We shall now show that the mapping I carries open sets in $D_n - Z - s_n$ into open sets. For a point in the interior of an E_k $(k \geq 2)$ which is not in Z, the Jacobian is positive, and I is therefore locally one-to-one; for a point in the interior of E_1, we have by construction a mapping which is locally one-to-one. Now let P be a point on the circle s_k $(2 \leq k \leq n - 1)$ which is not in Z. We know from the definition of F_k in (14) that F_j can be extended to a mapping $\tilde{F}_j$ by simply allowing ρ to vary from $j - 1 - \varepsilon$ to $j + \varepsilon$ $(\varepsilon > 0)$. It easily follows then that P is a point such that the Jacobian of $\tilde{F}_k$ is positive in an open set containing P. There is a circular neighborhood C with center P which is so small that it is mapped one-to-one by $\tilde{F}_k$ and by $\tilde{F}_{k+1}$, respectively. If the combined mapping ($\tilde{F}_k$ on E_k and $\tilde{F}_{k+1}$ on E_{k+1}) is not one-to-one on C, either $C \cap E_k$ or $C \cap E_{k+1}$ is mapped with its orientation reversed. But this contradicts the positiveness of the Jacobian in either $C \cap E_k$ or in $C \cap E_{k+1}$. On s_1, a similar argument holds.

Given now the mapping I_n: $D_n \to E^2$, let LM be the monotone-light factorization of I_n; for existence see [4, Theorem VIII, 4.1]. In this factorization, the monotone part of our particular map

$$M: D_n \to E^2$$

carries the closed disk D_n onto a topological disk $\bar{D}_n$. Since $M \mid \text{Bdry}(D_n)$ is one-to-one, there exists a homeomorphism h such that

$$h: \bar{D}_n \to D_n$$

and

$$h: M(p) = p \quad \text{for } p \in \text{Bdry}(D_n).$$

Hence, hM is a monotone mapping of D_n onto D_n which reduces to the identity mapping on the boundary. Furthermore, we have the mapping

$$Lh^{-1}: D_n \to E^2$$

and thus the monotone-light factorization

$$\bar{L}\bar{M} = (Lh^{-1})(hM).$$

Hence, we see that $\bar{M}$ maps D_n onto D_n, with the boundary fixed pointwise. We note that

(a) where I_n^* is not already defined or proved to be interior (I_n^* is defined interior in D_1 and proved to be interior on the s_k), the Jacobian is nonnegative;

(b) the set where the Jacobian vanishes does not separate D_n or $D_n - s_n$;

(c) the inverse $I_n^{*-1}(\pi)$ of each point π is compact.

Hence, if K is a component of a set $I_n^{*-1}(\pi)$, and $K \cap s_n = 0$, the argument of Theorem 1 in our paper [5] shows that on $D_n - \bar{M}^{-1}(\bar{M}(s_n))$, I_n^* is quasi-interior. By the argument of Theorem 3 in [5] the mapping $\bar{L}$ is interior on $D_n - s_n$, and

$$\bar{L} \mid s_n = I_n^* \mid s_n = \alpha.$$

Thus, given a curve α defined by the differential operator in (3), we have constructed the mapping I_n^* whose monotone-light factorization yields a light factor I which shows that α is an i-boundary. Thus the proof of Theorem 1 is complete.

5. Consider a curve β of the type described in Figure 1. Although β is of nonnegative circulation, its tangential winding number is zero, and hence β could not be the range of boundary values of a light interior mapping, and therefore not the range of boundary values of an analytic function. There also exist curves of nonnegative circulation for which the tangential winding number is one and which are not boundaries of a light interior mapping; for an example, see Figure 2. The problem of characterizing curves which are boundaries in a "purely topological" fashion has not been solved. Some progress in this direction is made in Theorem 2 of this paper.

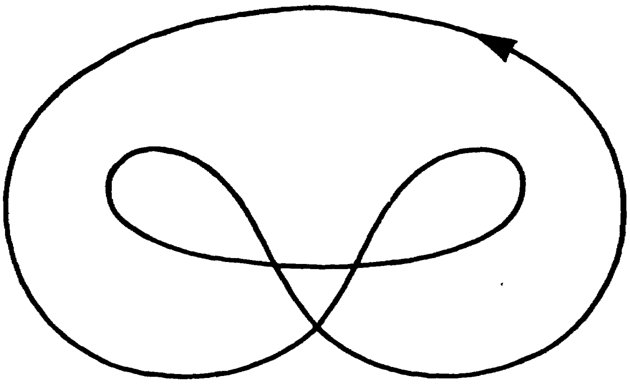

Figure 1

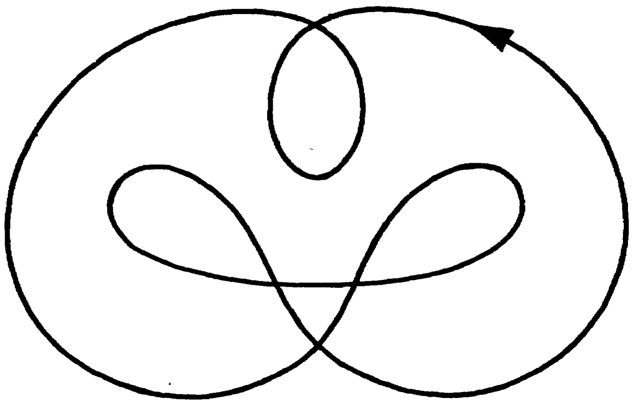

Figure 2

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