# TRANSIENT FLOWS IN NETWORKS

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## 1. INTRODUCTION

Ford and Fulkerson [1] have introduced the notion of dynamic flows in networks. A *dynamic network* consists of a graph $\Gamma$ to each edge e of which corresponds a nonnegative integer $\gamma(e)$, called the *capacity* of the edge, and a second nonnegative integer $\tau(e)$, called the *transit time* of the edge. In terms of transportation networks, the capacity $\gamma$ is to be thought of as giving an upper bound to the amount that can be shipped along an edge e, while the transit time $\tau$ specifies how long it takes a shipment to traverse this edge. In this framework, Ford and Fulkerson have considered the following problem: For a dynamic network $\Gamma$ with two distinguished terminals s and s' (called the *source* and the *sink*, respectively), to determine the maximum amount $\mu_k$ that can be shipped from s to s' in k time periods. In the work referred to, the authors describe an ingenious algorithm for obtaining $\mu_k$ for each integer k. More precisely, they show, for each integer k, how to obtain a *flow* $\phi_k$ (to be thought of as a shipping schedule) that achieves the desired shipment $\mu_k$ from s to s'.

Concerning the solution of Ford and Fulkerson, the following observation may be made. In order to achieve the maximum numbers $\mu_1, \mu_2, \cdots, \mu_k$, the authors construct a sequence of flows $\phi_1, \phi_2, \cdots, \phi_k$. It would be computationally advantageous if it turned out that $\phi_2$ is a "continuation" of $\phi_1$ and, in general, $\phi_{i+1}$ a continuation of $\phi_i$. Put another way, one might hope that the flow $\phi_k$ has the property that for each time $i < k$ the amount already shipped into s' is the maximum $\mu_i$. In this case the single flow $\phi_k$ would provide a solution to the maximum problem, not only for k time periods, but also for any smaller number of periods. However, the flows obtained by the authors do not have this desirable property; indeed, it is not clear from their work that such *universal maximal flows* exist. It is our purpose here to show that they exist, not only for the case treated by Ford and Fulkerson, but also for the considerably more general case in which the capacities $\gamma$ and transit times $\tau$ may vary with time.

## 2. A LEMMA ON STATIC NETWORK FLOWS

The result needed for proving the main theorem of this paper (see Section 3) is the Feasibility Theorem obtained by the author in [2]. We shall here record the definitions needed for a statement of that result. For motivation and interpretation of these definitions, the reader is referred to [2].

A *network with a source* is a triple $[X, s; \gamma]$, where X is a finite set of elements x, y, $\cdots$, called *nodes;* s is a distinguished node of X, called the *source;* and $\gamma$, the *capacity* of the network, is a function on pairs (x, y) of nodes, such that $\gamma(x, y)$ is a nonnegative integer or plus infinity.

A *flow* $\phi$ on X is a function from ordered pairs (x, y) to the integers satisfying the conditions

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$$\phi(x, y) + \phi(y, x) = 0 \qquad \text{(skew-symmetry)},$$

$$\phi(x, y) \leq \gamma(x, y) \qquad \text{(feasibility)}$$

(the skew-symmetry is simply the usual convention that the flow from x to y is the negative of the flow from y to x).

A *demand* $\delta$ is a function from X - s to nonnegative integers. The demand $\delta$ is called *feasible* if there exists a flow $\phi$ such that

$$\delta(y) \leq \sum_{x \in X} \phi(x, y)$$

for all y in X - s.

The main result of [2] states the following:

FEASIBILITY THEOREM. *The demand* $\delta$ *is feasible if and only if, for every subset* S *of* X - s, $\delta$ *satisfies the relation*

$$\sum_{y \in S} \delta(y) \leq \sum_{\substack{x \in X - S \\ y \in S}} \gamma(x, y). \tag{1}$$

We shall need a simple corollary of this theorem.

LEMMA 1. *Let* $y_1, \cdots, y_n$ *be distinct nodes of* X - s, *and let* $\delta_1, \cdots, \delta_n$ *be feasible demands such that*

$$\delta_i(y_i) = \mu_i \leq \delta_{i+1}(y_{i+1}) = \mu_{i+1}$$

*for* $i < n$. *Let* $\delta$ *be the demand such that*

$$\delta(y_1) = \mu_1,$$

$$\delta(y_i) = \mu_i - \mu_{i-1} \qquad \textit{for } i > 1,$$

$$\delta(y) = 0 \qquad \textit{otherwise}.$$

*Then the demand* $\delta$ *is feasible.*

*Proof.* Let S be any subset of X - s, and let k be the largest index for which $y_k$ belongs to S. Then

$$\sum_{y \in S} \delta(y) \leq \sum_{i \leq k} \delta(y_i) = \mu_k; \tag{2}$$

but since $\delta_k$ is feasible, it follows from (1) that

$$\mu_k = \delta_k(y_k) \leq \sum_{y \in S} \delta_k(y) \leq \sum_{\substack{x \in X - s \\ y \in S}} \gamma(x, y). \tag{3}$$

Combining (2) and (3), we see that $\delta$ satisfies (1) and is therefore feasible.

## 3. THE TRANSIENT-FLOW THEOREM

As mentioned in the Introduction, we intend to consider a generalization of the Ford-Fulkerson dynamic network in which capacities and transit times are allowed to vary with time. This generalization would seem to be useful in terms of applications. In rail networks, for example, it may happen that trains travel on some routes only on certain days of the week, so that the capacity of such a route is sharply increased on these occasions. Likewise, certain routes may sometimes be closed for periodic inspection or repair. We shall therefore give a slightly different formulation of a dynamic network from that of [1].

A *two-terminal network* $[X, s, s'; \gamma]$ is a network, in the sense of the preceding section, having an additional distinguished node $s'$, called the *sink*. For brevity, we shall henceforth denote this network simply by $X$.

A *maximal demand* $\delta$ on such a network is a feasible demand $\delta$ for which the value $\delta(s')$ is as large as possible.

Now, let $X$ be the set of nodes of a two-terminal network. We define $X_n$ to consist of all pairs $(x, i)$, where $x$ is in $X$ and $i \leq n$ is a nonnegative integer. For convenience, we denote such a pair by $x_i$.

An *n-stage two-terminal network* is a network $[X_n, s_0, s_n'; \gamma]$, where, as before, $\gamma$ is a function on pairs $(x_i, y_j)$ into nonnegative integers or infinity, which must also satisfy the condition

$$\gamma(s_i, s_{i+1}) = \gamma(s_i', s_{i+1}') = \infty. \tag{4}$$

For brevity, we henceforth denote this network by $X_n$.

An explanatory word concerning this definition is in order. The number $\gamma(x_i, y_j)$ gives an upper bound to the amount that can be shipped from node $x$ at time $i$ to arrive at node $y$ at time $j$. In view of this interpretation, one might expect the condition $\gamma(x_i, y_j) = 0$ for $j \leq i$ (the irreversibility of time). However, since our argument is independent of this condition, there is no reason to impose it. Condition (4) above states that goods can always be held over for any number of time periods at the source or sink. Notice that, in this formulation, no explicit mention is made of transit times. They are, however, implicitly included in the definition. Thus if edge $(x, y)$ has capacity 10 and transit time 3, this would be indicated by the relation

$$\gamma(x_i, y_j) = \begin{cases} 10 & \text{for } j = i + 3, \\ 0 & \text{otherwise.} \end{cases}$$

For the network $X_n$, the notions of flow, demand, feasible demand, and maximal demand are defined exactly as before.

Finally, for each integer $k \leq n$, we define $X_k$, the *k-stage subnetwork* of $X_n$, to be the two-terminal network $[X_k, s_0, s_k'; \gamma]$, where $\gamma$ is the same function as that for $X_n$, except that it is restricted to pairs $(x_i, y_j)$ with $i, j \leq k$.

LEMMA 2. *Let* $\delta_k$ *be a maximal demand on* $X_k$, *and let* $\mu_k = \delta_k(s_k')$. *Then* $\mu_k \leq \mu_{k+1}$ *for all* $k < n$.

*Proof.* Let $\phi_k$ be the maximal flow such that

$$\mu_k = \delta_k(s_k') = \sum_{x \in X_k} \phi_k(x, s_k').$$

Now define a new demand $\delta'$ as follows:

$$\delta'(x_i) = \delta_k(x_i) \qquad \text{for } x_i \in X_k,\ x_i \neq s_k',$$

$$\delta'(s_{k+1}') = \mu_k,$$

$$\delta'(x_i) = 0 \qquad \text{otherwise.}$$

This demand is feasible, since it is satisfied by choosing the flow to be $\phi'$, where

$$\phi'(x_i, y_j) = \phi_k(x_i, y_j) \qquad \text{for } i, j \leq k,$$

$$\phi'(s_k', s_{k+1}') = -\phi'(s_{k+1}', s_k) = \mu_k,$$

$$\phi'(x_i, y_j) = 0 \qquad \text{otherwise.}$$

By definition of a maximal demand, the conclusion of the lemma follows.

THEOREM. *Let* $\delta_1, \cdots, \delta_n$ *be maximal demands on* $X_1, \cdots, X_n$, *and let* $\mu_i = \delta_i(s_i')$. *Then the demand* $\delta$, *where* $\delta(s_1) = \mu_1$, $\delta(s_i) = \mu_i - \mu_{i-1}$ *for* $i > 1$, *and* $\delta(x) = 0$ *otherwise, is feasible.*

In view of Lemma 2, the theorem is simply a special case of Lemma 1, and the proof is therefore immediate.

## 4. REMARKS

(a) The problem of a universal maximal flow makes sense for the case where the network contains several sources $s_1, \cdots, s_n$. The analogous theorem is true in this case; namely, there exists a dynamic flow from the sources $s_1, \cdots, s_n$ into $s'$ that is maximal at all times $i \leq n$. In fact, this case is easily reduced to the case of a single source, by the standard device of introducing a new source $s_0$ into the network and defining the capacities $\gamma(s_0, s_i)$ to be infinite, and of transit time 1.

(b) One might hope that the theorem on universal maximal flows extends to the case where there is more than one sink. A natural generalization: let $\delta_1, \cdots, \delta_n$ be feasible demands on $X_1, \cdots, X_n$ such that $\delta_i(x_i) \leq \delta_{i+1}(x_{i+1})$ for all x; can one then prove the feasibility of the demand $\delta$ defined by these functions? The defining equations are

$$\delta(x_1) = \delta_1(x_1),$$

$$\delta(x_i) = \delta_i(x_i) - \delta_{i-1}(x_{i-1}) \qquad \text{for } i > 1.$$

The simple example illustrated in Figure 1 shows that this is not the case. Here the edges (s, x) and (x, y) both have capacities and transit times equal to 1. The graph of $X_2$ is shown in Figure 2. Now define $\delta_1$ and $\delta_2$ as follows:

$$\delta_1(x_1) = 1, \qquad \delta_1(y_1) = 0,$$

$$\delta_2(x_1) = 0, \qquad \delta_2(y_1) = 0, \qquad \delta_2(x_2) = 1, \qquad \delta_2(y_2) = 1.$$

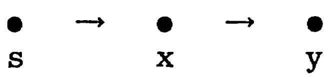


Fig. 1

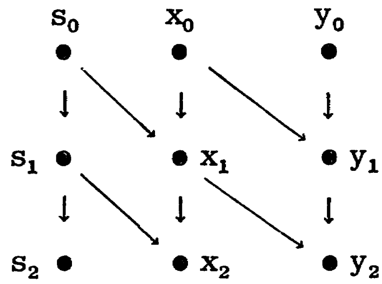


Fig. 2

Clearly, both $\delta_1$ and $\delta_2$ are feasible, but this is not the case for the demand $\delta$ defined by

$$\delta(x_1) = 1, \qquad \delta(y_1) = 0,$$

$$\delta(x_2) = 0, \qquad \delta(y_2) = 1,$$

as the reader will see on referring to Figure 2.

(c) For the case in which capacities are independent of time, Ford and Fulkerson showed that a dynamic maximal flow can be achieved that does not involve "hold-overs;" in our notation, the maximal flow $\phi_k$ has the property that $\phi_k(x_i, x_{i+1}) = 0$ for all $x_i$ other than s and s'. In view of their result, the same thing is true for the case of universal maximal flows when capacities are constant with time. Namely, one simply defines the capacity $\gamma$ so that $\gamma(x_i, x_{i+1}) = 0$ for $x \neq s, s'$, and the proof is exactly as before.

(d) For the case of constant capacities, Ford and Fulkerson have given a simple computational method for finding maximal flows. The proof given here does not lead to such a procedure. We suspect that some mild modification of the Ford-Fulkerson algorithm will actually allow the calculation of universal maximal flows. For the present, however, this is pure conjecture.

## REFERENCES

1. L. R. Ford, Jr. and D. R. Fulkerson, *Constructing maximal dynamic flows from static flows*, Operations Res. 6 (1958), 419-433.

2. D. Gale, *A theorem on flows in networks*, Pacific J. Math. 7 (1957), 1073-1082.

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