A NOTE ON A THEOREM OF H. KNOTHE

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In this journal, H. Knothe proved two "inverse Archimedean theorems" for convex bodies in Euclidean 3-space [1]. The present note extends the first of these two theorems and thereby furnishes a somewhat simpler proof of the first theorem itself.

Let C(u) be the cylinder whose generators have the direction u and which circumscribes the convex body K. Further, let B(u) be the breadth of K in the direction u, and L(u) the perimeter of the projection of K in the direction u. Since B(u) is continuous over the sphere of directions u, it has a maximum, say at u_0, and a minimum, say at u_1. Finally, denote by S(u) the lateral area of that part of C(u) which lies between the support planes of K in the directions u and -u.

THEOREM 1. *If* S(u) *is constant,* K *is of constant breadth.*

We first note that, for any u,

$$\pi B(u_0) \geq L(u) \geq \pi B(u_1)$$

because

$$L(u) = \pi \overline{B}(u), \tag{1}$$

where $\overline{B}(u)$ is the arithmetic mean of breadths orthogonal to u. Therefore

$$S(u_0) = B(u_0)\, L(u_0) \geq \pi B(u_0)\, B(u_1) \tag{2}$$

and

$$S(u_1) = B(u_1)\, L(u_1) \leq \pi B(u_1)\, B(u_0)\,. \tag{3}$$

Since

$$S(u_0) = S(u_1) = S(u)$$

for any direction u, we have, from the second parts of (2) and (3),

$$S(u) = \pi B(u_0)\, B(u_1)\,.$$

Then, by the first part of (2),

$$L(u_0) = \pi B(u_1)\,.$$

Because $B(u_1)$ is a minimum of B(u), it follows from (1) that

$$B(u) = B(u_1) \tag{4}$$

whenever u and u_0 are orthogonal.

Received December 6, 1957.

In a similar fashion we may conclude from the first part of (3) that

$$B(u) = B(u_0) \tag{5}$$

whenever u is orthogonal to u_1.

Now if u_0 and u_1 are coincident or opposite directions, K is plainly of constant breadth. If u_0 and u_1 are not coincident or opposite, let u' be a direction orthogonal to both u_0 and u_1. Then by (4) and (5)

$$B(u_1) = B(u') = B(u_0),$$

and so in this case also K is of constant breadth.

Call this constant value B. Let S(K) be the surface area of K, and s(u) the area of the projection of K in the direction u.

THEOREM 2 (Knothe). *If* S(u) = S(K), *then* K *is a sphere.*

By Theorem 1, K is of constant breadth B. The isoperimetric inequality applied to the projection of K in the direction u gives

$$\pi B^2/4 \geq s(u).$$

Application of Cauchy's formula, which gives S(K) as four times the arithmetic mean of s(u), yields

$$\pi B^2 \geq S(K),$$

with equality if and only if every projection of K is a circle, that is, if and only if K is a sphere. But by Theorem 1, if S(u) is constant, then

$$S(u) = \pi B^2 = S(K),$$

which shows that K is a sphere.

REFERENCE

1. H. Knothe, *Inversion of two theorems of Archimedes*, Michigan Math. J. 4 (1957), 53-56.

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