ORBITS OF UNIFORM DIMENSION

P. E. Conner

The purpose of this note is a study of topological transformation groups for which all the orbits have the same dimension. The investigation was suggested by a theorem of Montgomery and Samelson [7] to the effect that if a compact connected Lie group acts differentiably on the sphere, and if there is one stationary point, then the remaining orbits cannot all be of the same dimension. Some of our basic approach was suggested by Borel's note [1]. We have already made a preliminary study of this problem [3], and we shall amplify our methods here.

A transformation group $(\Gamma, X; m)$ is a triple consisting of a locally compact, arcwise connected, locally connected, finite-dimensional separable metric space X; a compact connected Lie group Γ; and a mapping $m: \Gamma \times X \to X$ satisfying

(i) $$m(e, x) = x,$$

(ii) $$m(g, m(h, x)) = m(gh, x).$$

As usual, we write $m(g, x) = gx$, and we suppress m, when we designate the transformation group. We introduce an equivalence relation into X: $x \sim y$ if and only if there is a $g \in \Gamma$ such that $gy = x$. The equivalence class of x, denoted by $O(x)$, is called the *orbit* of x; the decomposition space of X with respect to this equivalence relation is called the *orbit space*, and it will be denoted by X/Γ. We let

$$\eta: X \to X/\Gamma$$

be the natural projection onto the orbit space. For more details about these definitions, we refer to [9]. We shall make use of the fundamental notions of fibre bundles as found in [11]. Also, we shall refer to [2] for the definitions and techniques relevant to the use of spectral sequences. Throughout our note, $(L; X)$ will denote a transformation group with all orbits of the same dimension. When we wish to denote a more general type of transformation group, we use the letter Γ to denote the group acting. First, we shall collect some useful lemmas, and then make some applications of these at the end of our note.

For a transformation group (L, X), we denote by G_x, H_x and N_x, respectively, the isotropy group at x (the subgroup of L which leaves x fixed), the identity component of G_x, and the normalizer of H_x in L. Obviously, $H_x \subset G_x \subset N_x$. We consider a transformation group (L, X) in which all the orbits have the same dimension. Since all the orbits have the same dimension, $\dim H_x = \dim H_y$, but if x and y are sufficiently close, then there is an element $g \in L$ such that $gH_xg^{-1} \subset H_y$ [9]. Since H_x and H_y have the same dimension, $gH_xg^{-1} = H_y$. The space X is connected, and we conclude that the $\{H_x\}$ are all conjugate.

We denote by $F_x \subset X$ the set of all points in X which are stationary under H_x. If $F_x \cap F_y$ is not empty, then $H_x = H_y$; for otherwise the points in the intersection would be stationary under the closed subgroup generated by H_x and H_y; but this group contains $H_x \cup H_y$ in its identity component. There exists an element $g \in L$

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such that $gH_xg^{-1} \cup gH_yg^{-1} = H_x$, which means that $gH_xg^{-1} = H_x = gH_yg^{-1}$; therefore $H_x = H_y$.

If $g \in L$ is an element such that $gF_x = F_x$, then $g \in N_x$; for gF_x is the set of stationary points of gH_xg^{-1}. Let $\hat{F}_x$ denote the component of F_x containing x, and let B_x be the subgroup of elements mapping $\hat{F}_x$ onto itself; then $H_x \subset G_x \subset B_x \subset N_x$. We shall study the group B_x in detail. In particular, we shall find conditions under which B_x must be connected. We observe that the sets $\{g\hat{F}_x\}$ decompose X into disjoint closed sets; and we shall show that $\hat{F}_x$ actually fibres X. Let $\hat{B}_x = B_x/H_x$.

LEMMA 1. *The isotropy subgroups of the transformation group* $(\hat{B}_x, \hat{F}_x)$ *are all finite, and the orbit space* $\hat{F}_x/\hat{B}_x$ *is topologically* X/L.

The natural map η: X → X/L induces a map $\hat{\eta}$: $\hat{F}_x/\hat{B}_x \to X/L$ which is one-to-one onto, since $O(y) \cap \hat{F}_x = B_x(gy)$, where $g \in L$ is such that $gH_yg^{-1} = H_x$. We omit the proof that $\hat{\eta}^{-1}$ is continuous.

Now we shall use $\hat{F}_x$ to fibre X over L/B_x. We define a map

$$\tau: X \to L/B_x, \tag{1}$$

by sending the points in $g\hat{F}_x$ into the left coset gB_x, for all $g \in L$. Let ν: $L \to L/B_x$ be the natural map, and p: $L \times F_x \to L$ the projection; then we may factor τ in the commutative diagram

$$\begin{array}{ccc} L \times \hat{F}_x & \xrightarrow{m} & X \\ \rho \downarrow & & \downarrow \tau \\ L & \xrightarrow{\nu} & L/B_x. \end{array}$$

Since L is compact, the map m is closed and onto, which implies that τ is continuous. Since ν: $L \to L/B_x$ is a principal fibration, there exists a closed cell $J \subset L$ which is a cross section of ν at the identity. The cell J has the property that if $g, h \in J$ and $g^{-1}h \in B_x$, then $g = h$. Let us consider

$$m: J \times F_x \to X.$$

The image is closed, since m is a closed mapping; and if $gy = hz$, then $g^{-1}hz = y$, which implies that $g^{-1}h \in B_x$, so that $g = h$ and $y = z$. Therefore m is a homeomorphism which defines the local product structure in X. The translates $\{\nu(gJ)\}$ form a coordinate cover on L/B_x, for ν: $L \to L/B_x$ and for τ: $X \to L/B_x$. The coordinate transformations for $(L, L/B_x, B_x, \nu)$ determine those of $(X, L/B_x, \hat{F}_x, \tau)$ from the action of $\hat{B}_x$ on $\hat{F}_x$.

LEMMA 2. *For the transformation group* (L, X), *the map* (1) *defines a fibre bundle* $(X, L/B_x, F_x, \tau)$ *with structure group* $\hat{B}_x$.

We can use Lemma 2 to determine a simple condition which guarantees that B_x is connected.

LEMMA 3. *If for the transformation group* (L, X) *the space* X *is simply connected, then* B_x *is connected and equals the identity-component of* N_x. *Also*, $\pi_1(L/B_x) = 0$.

The homotopy sequence of $(X, L/B_x, F_x, \tau)$ immediately shows that $\pi_1(L/B_x) = 0$. Since

$g_1^{-1}h_1^{-1}hgy = y_1$, so that $g_1^{-1}h_1^{-1}hg \in B_x$, in other words, $h_1^{-1}h \in B_x$; and this implies that $h = h_1$ and $g_1^{-1}g \in G_x$. For any orbit of the form $hB_x(y)$ $(h \in J,\ y \in K_x)$, we have a factoring

$$\begin{array}{ccc} J\hat{B}_x K_x & \xrightarrow{r} & \hat{B}_x(x), \\ \uparrow i & \nearrow 1' & \\ h\hat{B}_x(y) & & \end{array}$$

where $1'^*: H^j(\hat{B}_x(x); Q) \simeq H^j(h\hat{B}_x(y); (Q)$. Clearly, the homomorphism

$$i^*: H^j(J\hat{B}_x K_x; Q) \to H^j(h\hat{B}_x(y); Q)$$

is onto.

We must show that the kernel of i^* is independent of h and y. We define

$$t(h,y): \hat{B}_x \to J \times \hat{B}_x \times K_x$$

by $t(y, h)(gH_x) = (h, gH_x, y)$. By homotopy, $t^*(h, y)m^*: H^j(J\hat{B}_x K_x; Q) \to H^j(\hat{B}_x; Q)$ is independent of h and y. We consider the commutative diagram

$$\begin{array}{ccc} \hat{B}_x & \xrightarrow{mt(h,\ y)} & J\hat{B}_x K_x, \\ 1' \nwarrow & & \nearrow i \\ & h\hat{B}_x(y) & \end{array}$$

and observe that 1^* is an isomorphism, so that the kernel of i^* is simply the kernel of $t^*(h, y)m^*$, which is indepent of h and of y. The map γ is regular, and $L/B_x \times X/$ is simply connected, so that the Leray sheaf determined by $H^j(\gamma^{-1}(x); Q)$ is constant.

We point out the following useful fact.

COROLLARY 1. *If for the transformation group* (L, X) *the subgroup* H_x *has maximal rank and* $\pi_1(X) = 0$, *then* $B_x = H_x$, *and* X *is topologically the Cartesian product* $L/B_x \times X/L$.

Since H_x is of maximal rank, the identity component of N_x is H_x, but by Lemma 2, $B_x = H_x$, and $\hat{B}_x = \{e\}$. This means that the map (3) $\gamma: X \to L/B_x \times X/L$ used in Theorem 1 is a homeomorphism.

Corollary 1 is useful in demonstrating that certain spaces do not support the action of a Lie group with all orbits of the same dimension. In particular, an open, simply connected manifold whose one-point compactification is again a manifold does not admit such a group of transformations [3].

We shall use Theorem 1 to prove a generalization of a theorem announced by Borel in [1]. For this, we shall use a corollary of the principal algebraic theorem of [2].

Let P denote a finite-dimensional vector space over the rationals which is graded by odd degrees; by $\wedge P$ we denote the exterior algebra generated by P. By A we denote a graded anticommutative algebra over Q, and we shall assume that $A^s = 0$ when s is large.

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THEOREM (Borel). *If* $\{E_r^{s,t}\}$ *is a canonical spectral sequence with*

$$E_2^{s,t} \simeq A^s \otimes (\wedge P)^t,$$

and if $E_\infty^i = 0$ *for all* i $(0 < i \neq n)$, *while* $E_\infty^n \simeq Q$, *then*

(a) $\wedge P$ *has a single odd-dimensional generator;*

(b) A^s *is a truncated polynomial ring on one generator.*

The generator of A^s is the image by transgression of the generator of $\wedge P$. The truncation occurs in dimension $n - r$, where r is the dimension of the generator of $\wedge P$. With this, we can immediately derive

THEOREM 2. *If the space* X *is simply connected and is a rational cohomology* n-*sphere, then each transformation group* (L, X) *satisfies one of the two conditions*

(i) X/L *is acyclic over the rationals;*

(ii) L *is either a circle group or a rational cohomology* 3-*sphere acting with finite isotropy groups.*

Since X is simply connected, we may use the spectral sequence (3) in Theorem 1. From the theorem of Borel, it follows that $\hat{B}_x$ is a rational cohomology r-sphere ($r = 1$ or $r = 3$) and that $H^*(L/B_x; Q) \otimes H^*(X/L; Q)$ is a truncated polynomial ring; hence either X/L or L/B_x is acyclic. The space L/B_x is orientable, since B_x is connected, so that if L/B_x is acyclic, then $L = B_x$ and H_x is normal in L. Since L is effective, $H_x = \{e\}$ and $L = B_x$.

In discussing local forms of these methods, we shall use the technique of [4]. Let $x \in X$, and define

$$I^j(x, X) \simeq \text{dir lim } H^j(V - x; Q),$$

where the direct limit is taken over all closed neighborhoods of x. This is a kind of local cohomology group. We shall say that X is Lc^1 at x if for each closed connected neighborhood V, there exists a $U \subset V$ such that $\pi_1(U - x) \to \pi_1(V - x)$ is trivial. If $x_0 \in X$ is a stationary point under Γ, then we identify x_0 with its image in X/Γ.

LEMMA 4. *If* $x_0 \in X$ *is a stationary point of* (Γ, X) *which satisfies the three conditions*

(i) *there is a closed connected invariant neighborhood of* x_0 *in which the remaining orbits are all of the same dimension;*

(ii) $I^j(x_0, X) \simeq H^j(S^n; Q)$ *for some* n;

(iii) X *is* Lc^1 *at* x_0;

and if Γ *is effective, then one of the following two alternatives holds:*

(a) $I^j(x_0, X/\Gamma) = 0 \quad (j > 0)$;

(b) Γ *is a circle group or a rational cohomology* 3-*sphere, operating with finite isotropy groups.*

We lose no generality by assuming that all orbits in X, other than x_0, have the same dimension. Let $\{U_i\}$ be a decreasing sequence of connected invariant neighborhoods of x_0 such that $\bigcap U_i = x_0$. Let $V_i = U_i - x_0$, and assume that $\pi_1(V_{i+1} - x_0) \to \pi_1(V_i - x_0)$ is trivial. Choose an $\hat{F}_x \subset X - x_0$ whose closure in X contains x_0, and let $F_i \subset V_i \cap \hat{F}_x$ be a decreasing sequence of closed connected sets with void intersection. Let $B_i \subset \Gamma$ be the subgroup of Γ which maps F_i onto itself. A slight modification of Lemma 3 shows that B_i is connected, and certainly $B_i \supset B_{i+1}$. For i large, $B_i = B_{i+1}$, and we shall take this to be the case. The space X/Γ is Lc^1 at x_0 by [10]. This construction leads to natural diagrams

$$\begin{array}{ccc} \gamma_i : V_i & \to & \Gamma/B_i \times V_{i+1}/\Gamma \\ \uparrow j & & \uparrow \tilde{j} \\ \gamma_{i+1} : V_{i+1} & \to & \Gamma/B_{i+1} \times V_{i+1}/\Gamma \end{array}$$

and it determines a direct limit system of spectral sequences, with

$$E_2^{s,t}(V_i) \simeq H^s(\Gamma/B_i \times V_i/\Gamma; Q) \otimes H^t(\hat{B}_i; Q),$$

whose E_∞-terms are associated with $H^*(V_i; Q)$. Passing to the direct limit as in [4], we have a spectral sequence, with

$$I_2^{s,t} \simeq \sum_{p+q=s} (H^p(L/B_i; Q) \otimes I^q(x_0, X/\Gamma)) \otimes H^t(\hat{B}_i; Q),$$

whose E_∞-term is associated with $I^*(x_0, X)$. The remainder of the argument follows the proof of Theorem 2.

It is known that when a group acts on a manifold, the orbits of highest dimension fill a dense open set [8]. We shall investigate the actions in which orbits of lower dimension are isolated. If (Γ, X) denotes a group action on a manifold, we let v be the dimension of the highest-dimensional orbit, and we let $S \subset X/\Gamma$ be the image under the natural map of the orbits $O(x)$ with $\dim O(x) < v$.

THEOREM 3. *Let* (Γ, X) *denote the operation of* Γ *on an orientable manifold, and assume that* S *consists of isolated points; then one of the following two alternatives holds:*

(i) $v = n - 1$;

(ii) *At each lower-dimensional orbit* $O(x_0)$, *there is a slice* K_{x_0} *on which* H_{x_0} *acts effectively either as a circle group or as a rational cohomology* 3-*sphere.*

Again, there is no loss of generality in assuming that there is only one orbit of lower dimension. We choose a slice K_{x_0}; it has dimension $n - \dim O(x_0)$. Under the natural map, K_{x_0} is mapped onto a neighborhood of x_0 in X/Γ. The group H_{x_0} acts on K_{x_0}, and $H_x \subset H_{x_0}$ for $x \in K_{x_0}$; thus the orbits of H_{x_0} acting on $K_{x_0} - x_0$ are all of the same dimension. As pointed out in [8], K_{x_0} is Lc^1 at x_0; furthermore, $I^j(x_0, K_{x_0}) \simeq H^j(S^{n-r-1}; Q)$. We now apply Lemma 4. If $I^j(x_0, X/\Gamma) = 0$ for $j > 0$, then, since $X/\Gamma - x_0$ is a generalized orientable $(n - v)$-manifold over the rationals, $n - v = 1$, so that $v = n - 1$. In the other case, H_x acts trivially on K_{x_0} for $x \neq x_0$. In particular, H_x is normal in H_{x_0} for $x \in K_{x_0}$.

THEOREM 4. *Let* (Γ, X) *denote the operation of a group on a manifold* X *in such a way that all of the lower-dimensional orbits are isolated; then the identity component of the isotropy group of the highest-dimensional orbits cannot be maximal unless* $v = n - 1$.

If the lower-dimensional orbits are removed from X, the resulting space is still connected, so that the identity components of the isotropy groups of orbits of dimension v are all conjugate. It is clear from Theorem 3 that if $v < n - 1$, these isotropy subgroups at the v-dimensional orbits cannot be maximal.

It should be pointed out that the case of a group acting on an n-manifold with an (n - 1)-dimensional orbit has been completely studied in [10].

COROLLARY 1. *If* (Γ, X) *denotes the action of a group on a closed manifold with a finite number of lower-dimensional orbits, and if* $v \neq n - 1$, *then the Euler characteristic of* X *is the sum of the Euler characteristics of the lower-dimensional orbits.*

COROLLARY 2. *If* (Γ, X) *denotes the action of a group on a closed manifold with negative Euler characteristic, and if* $v \neq n - 1$, *then there cannot be a finite number of lower-dimensional orbits.*

It is well known that the Euler characteristic of a homogeneous space is nonnegative [6].

COROLLARY 3. *If* (Γ, E^{2n+1}) *denotes a group acting on an odd-dimensional Euclidean space, and if* $v < 2n$, *there cannot be a finite number of lower-dimensional orbits.*

For the proof, we merely insert the point at infinity and apply Corollary 1.

COROLLARY 4. *If* (Γ, X) *denotes a group acting on a closed manifold with* $\chi(X) \neq 0$, *and if there are a finite number of lower-dimensional orbits, then there cannot be more than* $\chi(X)$ *lower-dimensional orbits.*

The case $v = n - 1$, is disposed of in [10]. If $v < n - 1$, then the isotropy group of some lower-dimensional orbit is maximal; thus, by Theorem 3, the isotropy group of every lower-dimensional orbit must be maximal, and therefore every lower-dimensional orbit has Euler characteristic at least 1.

For completeness, we include a result which proves the theorem of Borel that was announced in [1] and mentioned earlier in our note.

THEOREM 5. *If* (S^3, X) *denotes the effective action of the group of unit quaternions on a simply connected rational cohomology* n*-sphere* $(n \neq 3)$, *and if all isotropy groups are conjugate, then the isotropy group is the identity.*

Let us assume that the isotropy group G_x is nontrivial. The orbits $O(x) = S^3/G_x$ fibre X, and since X is simply connected, $\pi_1(S^3/G_x) \simeq G_x$ is abelian. The only abelian subgroups of S^3 are cyclic. The group $G_x \subset S^3$ cannot be normal, for the action is effective; thus the normalizer of G_x in S^3, denoted here by N_x, is the orthogonal group O(2). Let $\hat{F}_x \subset X$ be the component of the fixed point set of G_x which contains x, and let $B_x \subset N_x$ be the subgroup of N_x which maps $\hat{F}_x$ onto itself; then B_x is the rotation group R(2). As in Theorem 1, we can define a map

$$\gamma': X \rightarrow S^3/B_x \times X/S^3 . \tag{3'}$$

This determines a fibre bundle with fibre $\hat{B}_x = B_x/G_x$, which is again a circle group. There exists a spectral sequence

$$E_2^{s,t} \simeq H^s(S^2 \times X/S^3; Q) \otimes H^t(S^1; Q)$$

whose E_∞- term is associated with $H^*(X; Q)$. Since X is a cohomology n-sphere, and $n \neq 3$, this is a contradiction.

Note Added in Proof. In the last section, devoted to the localization of the earlier part, we add to the transformation group (Γ, x) the hypothesis that if K_x is a slice at an exceptional point x, then there is a second slice $K'_x \subset K_x$ such that the natural homomorphism $\pi_1(K'_x \setminus x) \to \pi_1(K_x \setminus x)$ is trivial.

REFERENCES

1. A. Borel, *Transformation groups with two classes of orbits*, Proc. Nat. Acad. Sci. U. S. A. 43 (1957), 983-985.
2. ———, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) 57 (1953), 115-207.
3. P. E. Conner, *On a theorem of Montgomery and Samelson*, submitted to Proc. Amer. Math. Soc.
4. ———, *On the impossibility of fibring certain manifolds by a compact fibre*, Michigan Math. J. 4 (2957), 241-247.
5. I. Fary, *Valeurs critiques et algèbres spectrales d'une application*, Ann. of Math. (2) 63 (1956), 437-490.
6. H. Hopf and H. Samelson, *Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen*, Comment. Math. Helv. 13 (1941), 240-251.
7. D. Montgomery and H. Samelson, *Fiberings with singularities*, Duke Math. J. 13 (1946), 51-56.
8. D. Montgomery and C. T. Yang, *The existence of a slice*, Ann. of Math. (2) (1957), 108-116.
9. D. Montgomery and L. Zippin, *Topological transformation groups*, New York, Interscience (1955).
10. P. S. Mostert, *On a compact Lie group acting on a manifold*, Ann. of Math. (2) 65 (1957), 447-455.
11. N. E. Steenrod, *The topology of fibre bundles*, Princeton University Press (1951).

University of Michigan