A REPRESENTATION THEORY FOR MEASURES ON BOOLEAN ALGEBRAS

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1. INTRODUCTION

In an abstract (L)-space with weak unit (see [3], [6]), the characteristic elements constitute a Boolean σ-algebra. The countably additive measures on this algebra may be identified with the elements of the given (L)-space. These elements, as they reappear in the second adjoint space, are usually identified with certain Baire measures on the zero-dimensional, compact Hausdorff space associated with the first adjoint space of the given (L)-space.

The principal purpose of this paper is to represent measures on an arbitrary Boolean algebra as Baire measures on a zero-dimensional, compact Hausdorff space. Once this purpose is achieved, certain applications of the representation will be indicated. Of these, the most important is intended to be a conversion process, wherein considerations of measures and measurable functions with respect to a σ-field of sets are replaced by considerations of Baire measures and continuous functions with respect to a zero-dimensional, compact Hausdorff space.

A first key device to be used is a decomposition of a measure into a countably additive portion and a purely finitely additive portion. This device seems to be due to M. Woodbury [9]. It has been used by E. Hewitt and K. Yosida [10] and by H. Bauer [1]. A second important technique is the development of a Baire measure from a content. Familiarity with the explanation of this technique given in [4] will be assumed. As a third aid, free use will be made of the theory of (L)-spaces as developed in [3] and [6].

2. PRELIMINARY CONCEPTS

Let $\mathfrak{B}$ be an abstract Boolean algebra. Let o and e denote the null element and the unit element in $\mathfrak{B}$, while $\bar{a}$ indicates the complement of the element a with respect to these elements. Let $a \vee b$ and $a \wedge b$ denote the lattice operations as applied to a pair of elements in $\mathfrak{B}$. Frequent use will be made of the symbol $\Pi_{n=1}^{\infty} a_n$ as denoting the greatest lower bound in $\mathfrak{B}$ of a sequence $\{a_n\}$ of elements, when such a bound exists.

Let ϕ denote a real-valued function defined on $\mathfrak{B}$. Consider the three following properties, which might be postulated for ϕ.

(I) $\sup_{a \in \mathfrak{B}} |\phi(a)| < \infty$;

(II) $\phi(a \vee b) = \phi(a) + \phi(b)$ for all $a, b \in \mathfrak{B}$ with $a \wedge b = o$;

(III) $\lim_{n \to \infty} \phi(a_n) = 0$ for each nonincreasing sequence $\{a_n\}$ with $\Pi_{n=1}^{\infty} a_n = o$.

A real-valued function ϕ defined on $\mathfrak{B}$ and possessing properties (I) and (II) is said to be a *measure* on $\mathfrak{B}$. A measure with the additional property (III) is called a

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countably additive measure, whereas a measure for which (III) fails for at least one such sequence is said to be *finitely additive.* A measure ϕ on $\mathfrak{B}$ is called *nonnegative* if $\phi(a) \geq 0$ for each element a of $\mathfrak{B}$, and *strictly positive* if $\phi(a) > 0$ for each element other than the zero element of $\mathfrak{B}$.

A nonnegative measure ϕ on $\mathfrak{B}$ is called *purely finitely additive* if the zero measure is the unique countably additive measure ψ on $\mathfrak{B}$ for which $0 \leq \psi(a) \leq \phi(a)$ for each $a \in \mathfrak{B}$. A measure ϕ on $\mathfrak{B}$ is called *purely finitely additive* if its disjoint, nonnegative components ϕ^+ and ϕ^-, in the usual lattice sense, are both purely finitely additive.

Now let ϕ be any measure on $\mathfrak{B}$. A decomposition of ϕ essential to our work is accomplished as follows.

First let ϕ^+, ϕ^- be defined by the relations

$$\text{(A)} \qquad \phi^+(a) = \sup_{b \leq a} \phi(b), \qquad -\phi^-(a) = \inf_{b \leq a} \phi(b),$$

for all $a \in \mathfrak{B}$. Next let ϕ^{+p} be defined by the relation

$$\text{(B)} \qquad \phi^{+p}(a) = \sup \left[\lim_n \{\phi^+(a_n)\} \mid a_n \leq a,\ a_{n+1} \leq a_n,\ \prod_{n=1}^{\infty} a_n = o \right]$$

for all $a \in \mathfrak{A}$. Let ϕ^{-p} be defined by a similar relation based on ϕ^-.

Finally let ϕ^{+c}, ϕ^{-c} be defined by the relations

$$\text{(C)} \qquad \phi^{+c}(a) = \phi^+(a) - \phi^{+p}(a), \qquad \phi^{-c}(a) = \phi^-(a) - \phi^{-p}(a),$$

for all $a \in \mathfrak{B}$. The analysis of ϕ may now be described.

THEOREM 2.1. *Let ϕ be a measure on $\mathfrak{B}$. Then each of the six functions ϕ^+, ϕ^-, ϕ^{+c}, ϕ^{-c}, ϕ^{+p}, ϕ^{-p} is a nonnegative measure on $\mathfrak{B}$, and*

$$\phi = [\phi^{+c} - \phi^{-c}] + [\phi^{+p} - \phi^{-p}]$$

is the (unique) expression of ϕ as the sum of a countably additive measure and a purely finitely additive measure.

Proof. Here ϕ^+, ϕ^- are familiar (see [4], [6]). There is little difficulty in verifying that ϕ^{+p}, ϕ^{-p} are nonnegative measures on $\mathfrak{B}$, and that the same is true of ϕ^{+c}, ϕ^{-c}. Moreover, ϕ^{+c}, ϕ^{-c} are countably additive on $\mathfrak{B}$. Thus let $\{a_n\}$ be a nonincreasing sequence of elements with $\Pi_{n=1}^{\infty} a_n = o$ in $\mathfrak{B}$. Note that

$$\phi^{+p}(a_m) \geq \lim_n \{\phi^+(a_n)\}$$

for all m. Then, from the relations

$$\phi^{+c}(a_n) + \phi^{+p}(a_n) = \phi^+(a_n) \qquad \text{and} \qquad \lim_n\{\phi^{+c}(a_n)\} + \lim_n\{\phi^{+p}(a_n)\} = \lim_n\{\phi^+(a_n)\},$$

one concludes that $\lim_n\{\phi^{+p}(a_n)\} = \lim_n\{\phi^+(a_n)\}$ and $\lim_n\{\phi^{+c}(a_n)\} = 0$. Thus ϕ^{+c} is countably additive on $\mathfrak{B}$, and a similar proof holds for ϕ^{-c}.

Finally, if ψ is a nonnegative, countably additive measure on $\mathfrak{B}$, with $0 \leq \psi(a) \leq \phi^{+p}(a)$ for each $a \in \mathfrak{B}$, it is to be shown that ψ is the zero measure on

$\mathfrak{B}$. For some $a \in \mathfrak{B}$, let $\psi(a) = 3r \geq 0$. Then $\phi^{+p}(a) \geq 3r$. By definition of ϕ^{+p}, there exists a nonincreasing sequence $\{a_n\}$ with $a_n \leq a$ and $\Pi_{n=1}^{\infty} a_n = o$ in $\mathfrak{B}$, with the property that $\lim_n\{\phi^+(a_n)\} \geq 2r$. Then $\phi^{+p}(a_n) \geq 2r$ for each a_n in this sequence, while $\lim_n \{\psi(a_n)\} = 0$. But $a = a_n \vee (a \wedge \bar{a}_n)$. Hence, for some subscript n_0, $\psi(a \wedge \bar{a}_{n_0}) \geq 2r$. But then $\phi^{+p}(a) \geq 4r$. Repeating this procedure, one would conclude that $\phi^{+p}(a) \geq nr$ for each positive integer n. However, ϕ (and thus ϕ^{+p}) is bounded on $\mathfrak{B}$. Hence $r = 0$ and ψ is the zero measure on $\mathfrak{B}$.

3. BOOLEAN MEASURES AS BAIRE MEASURES

Let $\mathfrak{B}$ continue to denote an abstract Boolean algebra. Let X_0 be the set of all measures on $\mathfrak{B}$ assuming the values 0 and 1 and no other values. For $a \in \mathfrak{B}$, let 0_a be the set of all $\phi \in X_0$ such that $\phi(a) = 1$. The sets 0_a being taken as a basis for open sets, X_0 is the zero-dimensional, compact, Hausdorff, Stone-representation space for $\mathfrak{B}$ [5]. Thus, under the correspondence $a \leftrightarrow 0_a$, the family of all open-closed subsets of X_0 is a faithful representation of $\mathfrak{B}$ as a Boolean algebra, attention being restricted to finite unions and intersections.

With respect to X_0, let $\mathfrak{M}$ be the class of all Baire sets, and let $\mathfrak{N}$ be the class of all Baire sets of the first category. Thus $\mathfrak{M}$ is the σ-algebra of subsets of X_0 generated by the compact G_δ-subsets of X_0 or, equivalently, the σ-algebra of subsets of X_0 generated by the open-closed subsets of X_0. A Baire measure on X_0 is understood, of course, to be a nonnegative, countably additive measure on $\mathfrak{M}$.

Every Baire measure ϕ_0 on X_0 is regular (see [4], p. 228). Thus

$$\phi_0(B) = \sup\{\phi_0(C) \mid C \subseteq B,\ C \text{ a compact } G_\delta\text{-set in } X_0\},$$

for each Baire set B of X_0. Since each compact G_δ-subset C of X_0 may be regarded as the set intersection $C = \bigcap_{n=1}^{\infty} 0_{a_n}$ of a nonincreasing sequence of open-closed subsets 0_{a_n}, one concludes that Baire measures on X_0 are uniquely specified by their values on the open-closed subsets of X_0.

Now let ϕ_0 be a Baire measure on X_0. Define a function ϕ on $\mathfrak{B}$ with $\phi(a) = \phi_0(0_a)$ according to the correspondence $a \leftrightarrow 0_a$. Then ϕ is a nonnegative measure on $\mathfrak{B}$. Conversely, let ϕ be a nonnegative measure on $\mathfrak{B}$. For each compact subset C of X_0, let $\lambda(C) = \inf\{\phi(a) \mid C \subseteq 0_a,\ a \in \mathfrak{B}\}$. Then the function λ is a content (see [4], p. 231), defined on the compact subsets of X_0. This content, in the manner described in [4], generates a Borel measure (and thus a Baire measure) ϕ_0 on X_0. In particular, for the open-closed subsets 0_a of X_0, one has

$$\phi(a) = \lambda(0_a) = \phi_0(0_a)$$

(see [4], p. 234, Theorem C). Therefore there exists a one-to-one onto correspondence of Baire measures on X_0 to nonnegative measures on $\mathfrak{B}$. This correspondence may be extended to a similar correspondence of signed Baire measures on X_0 to the signed measures on $\mathfrak{B}$.

Now let ϕ be a nonnegative measure on $\mathfrak{B}$. Let $\phi = \phi^c + \phi^p$ be an expression for it as the sum of a nonnegative, countably additive measure ϕ^c and a nonnegative, purely finitely additive measure ϕ^p on $\mathfrak{B}$. Let ϕ_0, ϕ_0^c, ϕ_0^p be the corresponding Baire measures. It is then easily verified that ϕ_0^c vanishes on every compact G_δ-set in X_0 with empty interior. Moreover, since every Baire set of the first category in X_0 has an empty interior, the measure ϕ_0^c is seen, by regularity, to be identically zero on the class $\mathfrak{N}$ of all Baire sets of the first category. Conversely, a Baire measure on X_0 vanishing on the class $\mathfrak{N}$ has value zero for all compact G_δ-sets with empty interior, these latter being nowhere-dense Baire subsets of X_0. Then the measure ϕ on $\mathfrak{B}$ corresponding to such a measure is seen to be countably additive on $\mathfrak{B}$, and one concludes that nonnegative, countably additive measures on $\mathfrak{B}$ are in one-to-one onto correspondence with Baire measures on X_0 vanishing on $\mathfrak{N}$.

Next, with Baire measure ϕ_0^p corresponding to ϕ^p, let

$$\psi_0(B) = \sup\{\phi_0^p(N) \mid N \subseteq B,\ N \in \mathfrak{N}\}$$

for each Baire set B of X_0. It is a small matter to verify that ψ_0, thus defined, is a Baire measure on X_0. Then $(\phi_0^p - \psi_0)$ is a nonnegative Baire measure on X_0, vanishing on the class $\mathfrak{N}$. If ψ is the corresponding countably additive measure on $\mathfrak{B}$, from $0 \leq \psi \leq \phi^p$, one concludes that ψ is the zero measure on $\mathfrak{B}$, and thus that $(\phi_0^p - \psi_0)$ is the zero Baire measure. Thus $\phi_0^p = \psi_0$. Here the construction shows that $\phi_0^p = \psi_0$ is zero outside a certain Baire set of the first category. Conversely, it is easily seen that a Baire measure on X_0 that is zero outside a Baire set of the first category determines a purely finitely additive measure on $\mathfrak{B}$.

Finally, for the Baire measure ϕ_0 and for each Baire set B, let

$$\psi_0^p(B) = \sup\{\phi_0(N) \mid N \subseteq B,\ N \in \mathfrak{N}\}.$$

Let $\psi_0^c = \phi_0 - \psi_0^p$, so that $\phi_0 = \psi_0^c + \psi_0^p$. Again it is easily verified that ψ_0^p is a Baire measure vanishing outside a certain Baire set of the first category, while ψ_0^c is a Baire measure vanishing on $\mathfrak{N}$. Then, with $\phi_0 = \phi_0^c + \phi_0^p = \psi_0^c + \psi_0^p$, it is a small matter to conclude that $\phi_0^c = \psi_0^c$ and $\phi_0^p = \psi_0^p$. Thus the decomposition $\phi = \phi^c + \phi^p$ of a nonnegative measure ϕ on $\mathfrak{B}$ into a sum of countably and purely finite additive portions in unique.

THEOREM 3.1. *The measures on an abstract Boolean algebra* $\mathfrak{B}$ *are in one-to-one onto correspondence to the Baire measures on the Stone-representation space* X_0 *of* $\mathfrak{B}$. *A measure on* $\mathfrak{B}$ *is countably additive if and only if its Baire counterpart vanishes on all Baire sets of the first category. A measure on* $\mathfrak{B}$ *is purely finitely additive if and only if its Baire counterpart vanishes outside a Baire set of the first category.*

4. BOOLEAN MEASURES AS ABSTRACT (L)-SPACES

As is well known [7], the linear, normed lattice of all Baire measures on a compact Hausdorff space is an abstract (L)-space. Under the correspondence just developed, the set of all measures on a Boolean algebra is then also an abstract (L)-space. However, in view of the decomposition of a Baire measure into a Baire measure identically zero on $\mathfrak{N}$ and a Baire measure zero outside a Baire set $N \in \mathfrak{N}$, together with the obvious disjointness of any two Baire measures of these distinct types, one concludes that the abstract (L)-space of all Baire measures may be viewed as the cross product of two abstract (L)-spaces, one composed of all signed Baire measures identically zero on $\mathfrak{N}$, and the other composed of all signed Baire measures vanishing outside some element of $\mathfrak{N}$. This cross product of Baire measures is reflected in a cross product of the countably additive and the purely finitely additive measures of an arbitrary abstract Boolean algebra.

THEOREM 4.1. *For arbitrary abstract Boolean algebra* $\mathfrak{B}$, *the set of all measures, the set of all countably additive measures and the set of all purely finitely additive measures each constitute an abstract* (L)-*space. If these three sets of measures are regarded as abstract* (L)-*spaces, the first is the cross product of the remaining two.*

An interesting application of this viewpoint is found in the theory of abstract (L)-spaces with weak unit. In general, such spaces are not reflexive [7]. However, in such spaces the set of all characteristic elements constitutes a Boolean σ-algebra. Moreover, the given abstract (L)-space can be identified with the abstract (L)-space of all countably additive measures on this σ-algebra [3]. At the same time, the first adjoint space is the (M)-space of all continuous, real-valued functions on the compact Hausdorff space that is the Stone-representation space for this Boolean σ-algebra. Finally, the second adjoint space is the (L)-space of all Baire measures on this zero-dimensional, compact Hausdorff space. On expressing this second adjoint space of Baire measures as a cross product in the manner indicated above, the first factor (that is, the factor composed of all Baire measures vanishing identically on $\mathfrak{N}$) can be identified with the given abstract (L)-space.

THEOREM 4.2. *The second adjoint space of an abstract* (L)-*space with weak unit is the cross product of the given abstract* (L)-*space together with a second, newly introduced abstract* (L)-*space.*

5. STRICTLY POSITIVE BOOLEAN MEASURES

Henceforth, let $\mathfrak{B}$ denote a Boolean σ-algebra. The conditions under which such algebras possess a strictly positive, countably additive measure have been much discussed. This section is intended to add one comment to that discussion. If there exists a strictly positive, countably additive measure on $\mathfrak{B}$, then clearly any family of pairwise disjoint elements of $\mathfrak{B}$ is at most countably infinite. More interesting, however, is the fact that this condition suffices to distinguish an element a_o in $\mathfrak{B}$, with the property that there exists a nonnegative, countably additive measure on $\mathfrak{B}$ which is strictly positive on elements of $\mathfrak{B}$ contained in a_o, while every countably additive measure on $\mathfrak{B}$ is identically zero on elements of $\mathfrak{B}$ contained in the complement of a_o.

Thus, let $\mathfrak{B}$ be a Boolean σ-algebra satisfying the stated countability condition. Let ϕ be any nonnegative, countably additive measure on $\mathfrak{B}$. By Zorn's lemma; one may form a maximal family of pairwise disjoint elements of $\mathfrak{B}$ on which ϕ vanishes. Under the assumed condition, this family is at most countable. Under the σ-additivity condition, ϕ vanishes on the element of $\mathfrak{B}$ which is the union in $\mathfrak{B}$ of the elements of this family. Because of the maximality of the family, ϕ vanishes on no element of $\mathfrak{B}$ contained in the complement of this union. Thus to each such measure ϕ is assigned an element of $\mathfrak{B}$ with the property that ϕ is strictly positive on elements of $\mathfrak{B}$ contained in this element, and identically zero on elements of $\mathfrak{B}$ contained in the complement of this element.

Next, let ψ be a second nonnegative and countably additive measure on $\mathfrak{B}$. Assume that $\phi \wedge \psi = o$. Let a_o represent the intersection in $\mathfrak{B}$ of the elements assigned as above to ϕ and ψ. Recall that $[\phi \wedge \psi](a_o) = \text{g.l.b.}_{b \leq a_o}\{\phi(b) + \psi(\bar{b} \wedge a_o)\}$. By virtue of this fact and under the assumption that $\phi \wedge \psi = o$ one may construct, for arbitrary $\varepsilon > 0$, a nonincreasing sequence $\{b_n\}$ with $b_n \leq a_o$, with $\phi(b_n) \leq \varepsilon/2^n$ and $\psi(\bar{b}_{n+1} \wedge b_n) \leq \varepsilon/(2^{n+1})$, while $\psi(\bar{b}_1 \wedge a_o) \leq \varepsilon/2$. Then $\phi(\Pi_{n=1}^{\infty} b_n) = 0$. This implies that $\Pi_{n=1}^{\infty} b_n = o$, since ϕ is strictly positive for elements of $\mathfrak{B}$ contained in a_o.

Hence $a_o = (a_o \wedge \bar{b}_1) \vee (b_1 \wedge \bar{b}_2) \vee \cdots$. This, however, means that $\psi(a_o) \leq \varepsilon$. Thus $a_o = o$ in $\mathfrak{B}$, since ψ likewise is strictly positive on nonzero elements of $\mathfrak{B}$ contained in a_o. Hence, if ϕ and ψ are any two nonnegative, countably additive measures on $\mathfrak{B}$, with the property that $\phi \wedge \psi = o$, then the elements of $\mathfrak{B}$ assigned as above to ϕ and ψ are disjoint.

Finally, let $\{\phi_\gamma, \gamma \in \Gamma\}$ denote a maximal family of pairwise disjoint, nonnegative, countably additive measures on $\mathfrak{B}$. Since the elements of $\mathfrak{B}$ associated as above with these measures form a family of pairwise disjoint elements of $\mathfrak{B}$, it follows that this family of elements of $\mathfrak{B}$, and thus the given maximal family of measures, is at most countable. Let $\{\phi_n\}$ $(n = 1, 2, \cdots)$ denote this family, according to some enumeration, after each measure has been normalized with value 1 on the unit element in $\mathfrak{B}$. Then $\phi = \Sigma_{n=1}^{\infty} (1/2^n) \cdot \phi_n$ is a countably additive measure that is strictly positive for elements of $\mathfrak{B}$ contained in the union of the elements assigned to the individual ϕ_n, and it vanishes identically on elements of $\mathfrak{B}$ contained in the complement of this union. Moreover, by the assumed maximality of the family of measures, the trivial zero measure is the only countably additive measure possible on this complementary element. This may be summarized as follows.

THEOREM 5.1. *In any Boolean σ-algebra $\mathfrak{B}$, with the property that each family of pairwise disjoint elements of the algebra is at most countably infinite, there exists an element of $\mathfrak{B}$ and a countably additive measure on $\mathfrak{B}$ such that this measure is strictly positive on elements of $\mathfrak{B}$ contained in this element, while this measure and every other countably additive measure on $\mathfrak{B}$ is identically zero on elements of $\mathfrak{B}$ contained in the complement of this element.*

Each element of a Boolean σ-algebra determines a Boolean σ-algebra composed of all elements majorized by that element (such an algebra will be called a *principal* subalgebra). Moreover, any set of necessary and sufficient conditions for the existence of a strictly positive, countably additive measure must be equally applicable to the entire algebra and to each principal subalgebra. Hence, the preceding theorem indicates that it would be of equal profit to seek a set of conditions, applicable at once to the whole algebra and to each principal sub-algebra, which are necessary and sufficient for the existence of some nontrivial, countably additive measure.

6. MEASURABLE SPACES AND BAIRE MEASURES

Let $\mathfrak{M}$ denote a σ-field of subsets of a set Y. Assume that $\mathfrak{M}$ is *separating* in the sense that, for each pair of distinct points p and q in Y, there is an element E in $\mathfrak{M}$ with $p \in E$ and $q \notin E$.

The elements of $\mathfrak{M}$, as partially ordered by the inclusion relation, constitute a Boolean σ-algebra. Measures on $\mathfrak{M}$ may be considered as measures on this algebra, and conversely. The distinction of countably additive and purely finitely additive measures, along with the decomposition procedures, is preserved in this conversion. In short, all measure concepts introduced for Boolean σ-algebras are now transferred unchanged to this σ-field of sets.

Let $X(\mathfrak{M})$ denote the Stone-representation space of $\mathfrak{M}$ considered as a Boolean algebra. The points of $X(\mathfrak{M})$ may be regarded as zero-one measures on $\mathfrak{M}$. They may also be considered as prime dual ideals of elements of $\mathfrak{M}$ viewed as a Boolean algebra. Each point in the set Y may be identified with a unique point in $X(\mathfrak{M})$, so that Y may be regarded as a subset of $X(\mathfrak{M})$. The residues in Y of the open-closed subsets of $X(\mathfrak{M})$ are the subsets of Y in the field $\mathfrak{M}$.

The set Y, as underlying the σ-field $\mathfrak{M}$, is not assumed to possess a topology. However, even though the continuity concept is inapplicable, the concept of a real-valued function, defined on Y and measurable with respect to $\mathfrak{M}$, is available. Now let $\bar{f}$ denote a real-valued function defined and continuous on $X(\mathfrak{M})$, and let f denote the restriction of $\bar{f}$ on $X(\mathfrak{M})$ to the subset Y of $X(\mathfrak{M})$.

THEOREM 6.1. *The functions* $\bar{f}$ *defined and continuous on* $X(\mathfrak{M})$, *as restricted to the subset* Y *of* $X(\mathfrak{M})$, *are identical with the functions* f *defined and bounded on* Y, *and measurable with respect to* $\mathfrak{M}$.

Proof. Any function $\bar{f}$, defined and continuous on $X(\mathfrak{M})$, is Baire measurable. Thus, for each real number r, the set $P(\bar{f}, r) = \{p \in X(\mathfrak{M}) \mid \bar{f}(p) > r\}$ is a Baire subset of $X(\mathfrak{M})$. Hence (see [4], p. 223), $P(\bar{f}, r)$ is congruent to an open-closed subset 0_E of $X(\mathfrak{M})$ modulo a Baire subset N of the first category. Such a set N, however, is a Baire subset of a countable union $\bigcup_{n=1}^{\infty} C_n$, where each C_n is a compact G_δ-subset of $X(\mathfrak{M})$ with empty interior. Thus $C_n = \bigcap_{m=1}^{\infty} 0_{E_{nm}}$, where each $0_{E_{nm}}$ is an open-closed subset of $X(\mathfrak{M})$, determined by an element E_{nm} of $\mathfrak{M}$. If a point p_o of the subset Y of $X(\mathfrak{M})$ were in such a C_n, then it would follow that $p_o \in E_o = \bigcap_{m=1}^{\infty} E_{nm}$, and the set E_o would be a nonzero element of $\mathfrak{M}$. Then $0_{E_o} \subseteq C_n$, contrary to the assumption that C_n had an empty interior. Thus each C_n is without points of the subset Y of $X(\mathfrak{M})$ and, in consequence, so likewise is the set N. Therefore, with $P(\bar{f}, r)$ congruent to 0_E modulo N in $X(\mathfrak{M})$, one concludes that $P(\bar{f}, r) \cap Y = E$. Hence, with f denoting $\bar{f}$ as restricted to Y, the set $P(f, r) = \{p \in Y \mid f(p) > r\}$ is seen to be an element of $\mathfrak{M}$. Thus f is a function defined and bounded on Y, measurable with respect to the σ-field $\mathfrak{M}$.

The converse statement, that each function f, defined and bounded on Y and measurable with respect to $\mathfrak{M}$, determines a (unique) function $\bar{f}$ defined and continuous on $X(\mathfrak{M})$ whose restriction to Y is identical with the given f, is easily established and need not delay us.

Now let $L(Y, \mathfrak{M})$ denote the space of all functions f defined and bounded on Y and measurable with respect to $\mathfrak{M}$. To each element f of $L(Y, \mathfrak{M})$, assign the norm $\|f\| = \sup\{|f(p)| \mid p \in Y\}$. With this norm, $L(Y, \mathfrak{M})$ becomes a Banach space. This space, as the preceding theorem indicates, is equivalent to the Banach space $C[X(\mathfrak{M})]$ of continuous functions on $X(\mathfrak{M})$, with the usual norm. The bounded linear functionals F_0 on this latter space are, of course, determined by Baire measures ϕ_0 on $X(\mathfrak{M})$, and they assume the form $F_0(\bar{f}) = \int_X \bar{f}\, d\phi_0(x)$. Such measures ϕ_0, however, through the relation $\phi(E) = \phi_0(0_E)$, are in one-to-one onto correspondence to measures ϕ on the σ-field $\mathfrak{M}$. Thus the bounded linear functionals on $L(Y, \mathfrak{M})$ are all of the form $\int_Y f\, d\phi(y)$, where ϕ is any measure on the field $\mathfrak{M}$.

THEOREM 6.2. *Measures on the field* $\mathfrak{M}$ *and bounded linear functionals on the space* $L(Y, \mathfrak{M})$ *are in one-to-one onto correspondence to Baire measures on* $X(\mathfrak{M})$ *and bounded linear functionals on* $C[X(\mathfrak{M})]$ *through the relations* $\phi(E) = \phi(0_E)$ *and* $F(f) = \int_Y f\, d\phi = \int_X \bar{f}\, d\phi_0 = F_0(\bar{f})$.

We next introduce a σ-ideal $\mathfrak{N}$ of elements of the σ-field $\mathfrak{M}$ of subsets of the set Y. Let $L(Y, \mathfrak{M})$ continue to denote the space of functions defined and bounded on Y, and measurable with respect to $\mathfrak{M}$. Now, however, to each element f of $L(Y, \mathfrak{M})$, assign as norm $\|f\|_\infty$ the infimum of the real numbers r with the property that the elements $E(r) = \{p \in Y \mid |f(p)| > r\}$ of $\mathfrak{M}$ are in the ideal $\mathfrak{N}$. Finally, after identifying elements f and g of $L(Y, \mathfrak{M})$ with $\|f - g\|_\infty = 0$, let $L_\infty(Y, \mathfrak{M}, \mathfrak{N})$ denote the resulting Banach space.

Continue to let $X(\mathfrak{M})$ have its earlier significance. Now, however, let $X(\mathfrak{N})$ denote the subspace of $X(\mathfrak{M})$ consisting of all points representing zero-one measures on $\mathfrak{M}$ vanishing on $\mathfrak{N}$ or, equivalently, representing prime dual ideals of the algebra $\mathfrak{M}$ containing no elements of the ideal $\mathfrak{N}$. In terms of these descriptions, it is clear that $X(\mathfrak{N})$ is a closed subset of $X(\mathfrak{M})$, and thus a zero-dimensional, compact Hausdorff space. It is easily verified that the open-closed subsets of $X(\mathfrak{N})$ are all of the form $0_E \cap X(\mathfrak{N})$, where E is an element of $\mathfrak{M}$. It should be noted, however, that a single open-closed subset of $X(\mathfrak{N})$ may be derived from many different elements of $\mathfrak{M}$, while each element of $\mathfrak{M}$ in $\mathfrak{N}$ corresponds to the empty set in $X(\mathfrak{N})$.

It will now be shown that the Banach space $L_\infty(Y, \mathfrak{M}, \mathfrak{N})$ is equivalent to the Banach space $C[X(\mathfrak{N})]$. Thus let f be any element of $L(Y, \mathfrak{M})$, and let $\bar{f}$ be the corresponding continuous function on $X(\mathfrak{M})$, as explained earlier. We wish to show that $\|f\|_\infty = \sup\{|\bar{f}(p)| \mid p \in X(\mathfrak{N})\}$, so that $\|f\|_\infty$ equals the uniform norm of $\bar{f}$ as restricted to $X(\mathfrak{N})$. Denote this restriction of $\bar{f}$ by $\bar{f}_o$. Also, to avoid absolute-value signs, we assume f and $\bar{f}$ to be nonnegative. Let r be any real number greater than $\|f\|_\infty$. Let $F(r) = \{p \in X(\mathfrak{M}) \mid \bar{f}(p) \geq r\}$. If $p_o \in F(r)$, then the prime dual ideal of elements of $\mathfrak{M}$ represented by p_o must contain an element of $\mathfrak{M}$ in the ideal $\mathfrak{N}$. Hence each such subset F(r) of $X(\mathfrak{M})$ is disjoint from the subspace $X(\mathfrak{N})$. Thus $\|\bar{f}_o\| \leq \|f\|_\infty$. Next let r be any number less than $\|f\|_\infty$, and let F(r) be as above. F(r) is a compact subset of $X(\mathfrak{M})$. If F(r) were disjoint from $X(\mathfrak{N})$, it would be possible to cover F(r) in $X(\mathfrak{M})$ by open-closed subsets of the type 0_E, where each E is an element of $\mathfrak{M}$ in $\mathfrak{N}$. Then, in virtue of the compactness of F(r), there would be a single element E_o of $\mathfrak{M}$ in $\mathfrak{N}$, with the property that 0_{E_o} contains F(r). This, however, would imply that $\|f\|_\infty = r < \|f\|_\infty$. Thus, the sets F(r) must contain points of $X(\mathfrak{N})$ for each real number r less than $\|f\|_\infty$. The desired conclusion, that $\|f\|_\infty = \|\bar{f}_o\|$, is now established.

It is now clear that each element f of $L(Y, \mathfrak{M})$ determines an element $\bar{f}_o$ of $C[X(\mathfrak{N})]$ with $\|f\|_\infty = \|\bar{f}_o\|$. Also, two elements f and g of $L(Y, \mathfrak{M})$ are to be identified in $L_\infty(Y, \mathfrak{M}, \mathfrak{N})$ (with $\|f - g\|_\infty = 0$) if and only if $\bar{f}_o = \bar{g}_o$ on $X(\mathfrak{N})$. Finally, it should be shown that every element h of $C[X(\mathfrak{N})]$ has the property that $h = \bar{f}_o$ for some element f of $L(Y, \mathfrak{M})$. This is easily done in virtue of the Stone-Weierstrass approximation theorem, once one adverts to the fact that, with each element f of $L(Y, \mathfrak{M})$ with corresponding element $\bar{f}_o$ of $C[X(\mathfrak{N})]$, there is associated an element g of $L(Y, \mathfrak{M})$ with $\|g\| = \|g\|_\infty$ and $\bar{g}_o = \bar{f}_o$. We note in passing that this entire last development is equivalent to the statement that every function defined and continuous on $X(\mathfrak{N})$ has a continuous extension on $X(\mathfrak{M})$.

It has now been established that the Banach space $L_\infty(Y, \mathfrak{M}, \mathfrak{N})$ is equivalent to the Banach space $C[X(\mathfrak{N})]$. Thus the bounded, linear functionals on $L_\infty(Y, \mathfrak{M}, \mathfrak{N})$ correspond to those on $C[X(\mathfrak{N})]$, and thence to Baire measures on $X(\mathfrak{N})$. If ϕ_0 is a measure of this latter type, then the function ϕ, defined on $\mathfrak{M}$ through the relation $\phi(E) = \phi_0(0_E \cap X(\mathfrak{N}))$, is a measure defined on $\mathfrak{M}$ and vanishing on $\mathfrak{N}$. Conversely, if ϕ is a measure on $\mathfrak{M}$ vanishing on $\mathfrak{N}$, then, for elements E and F of $\mathfrak{M}$ with $0_E \cap X(\mathfrak{N}) = 0_F \cap X(\mathfrak{N})$, it is easily shown that $\phi(E) = \phi(E \cap F) = \phi(F)$. From this it follows that there is an unique Baire measure ϕ_0 on $X(\mathfrak{N})$ with

$$\phi(E) = \phi_0(0_E \cap X(\mathfrak{N})).$$

It is now established that there is a one-to-one onto correspondence of measures on $\mathfrak{M}$ vanishing on $\mathfrak{N}$ to Baire measures on $X(\mathfrak{N})$. Moreover, there is no difficulty in showing, first, that the countably additive measures on $\mathfrak{M}$ vanishing on $\mathfrak{N}$ correspond to Baire measures on $X(\mathfrak{N})$ vanishing on Baire sets of the first category, and second, that the purely finitely additive measures on $\mathfrak{M}$ vanishing on $\mathfrak{N}$ correspond to Baire measures identically zero outside a Baire set of the first category. Finally, with measures ϕ and ϕ_0 corresponding as described above, and bounded linear functional F on $L_\infty(Y, \mathfrak{M}, \mathfrak{N})$ corresponding to F_0 on $C[X(\mathfrak{N})]$, one concludes that

$$F(f) = \int_Y f\, d\phi = \int_{X(\mathfrak{N})} \bar{f}_0\, d\phi_0 = F_0(\bar{f}_0).$$

These results are now summarized.

THEOREM 6.3. *Through the relationship* $[f] \leftrightarrow \bar{f}$ *with* $\|f\|_\infty = \|\bar{f}_0\|$, *the Banach spaces* $L_\infty(Y, \mathfrak{M}, \mathfrak{N})$ *and* $C[X(\mathfrak{N})]$ *are made equivalent. The resulting equivalence of their adjoint spaces shows itself in the convertibility of measures on* $\mathfrak{M}$ *vanishing on* $\mathfrak{N}$ *with Baire measures on* $X(\mathfrak{N})$.

The developments of this section are, essentially, a reformulation of work found in [10]. However, the direct conversion of measurable functions into continuous functions and of set measures into Baire measures, here employed, provides notable economy of concept, proof and understanding. It may now be noted that the final conjecture of [10] is substantially correct. A correct version of the conjecture follows.

THEOREM 6.4. $L_\infty^*(Y, \mathfrak{M}, \mathfrak{N})$ *contains no countably additive measure if and only if every counterpart Baire measure can be confined to a Baire set of the first category.*

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