ON SOLUTIONS OF THE EQUATION OF HEAT CONDUCTION

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1. INTRODUCTION

Suppose that $u = u(x, t)$ is defined over some domain D in the xt-plane. We say that u is parabolic in D if u is of class C^2 and $u_t = u_{xx}$ in D. Parabolic functions have many properties in common with harmonic functions. In this paper we use analogues of well-known theorems on harmonic and subharmonic functions to obtain a uniqueness theorem and some representation theorems for functions which are parabolic in the infinite strip $0 < t < c$.

We begin by introducing a class of subparabolic functions. For $x_1 < x_2$ and $t_1 < t_2$, let $R = R(x_1, x_2; t_1, t_2)$ denote the open rectangle bounded by the lines $x = x_1$, $x = x_2$, $t = t_1$, and $t = t_2$. Let S denote that part of the boundary for R which does not lie in the line $t = t_2$, and let $w = w(x, t)$ denote any function continuous in a domain D. When $R \cup S \subset D$, we define the function $M_R\, w = M_R\, w(x, t)$ as follows:

$$(1.1)\qquad \begin{cases} M_R\, w = w & \text{in } D - R, \\ M_R\, w & \text{is parabolic in } R, \\ M_R\, w & \text{is continuous at each point of } S. \end{cases}$$

Finally, we say that w is subparabolic in D if $w \leq M_R\, w$ for each rectangle R ($R \cup S \subset D$). (Compare [9].) When $P_0 = (x_0, t_0) \in R$, we have

$$(1.2)\qquad M_R\, w(P_0) = \int_S G(P_0, Q)\, w(Q)\, dS\,,$$

where the integration is taken over S, where

$$(1.3)\qquad \int_S G(P_0, Q)\, dS = 1\,,$$

and where

$$(1.4)\qquad G(P_0, Q) \geq 0$$

for $Q = (\xi, \tau) \in S$ [6]. Excepting the corners, we have strict inequality in (1.4) if and only if $t_1 \leq \tau \leq t_0$.

If w is subparabolic in D, then w is subparabolic in each subdomain of D. If u is parabolic, then $|u|^p$ is subparabolic for $p \geq 1$. A function u is parabolic if and only if the functions u and -u are subparabolic. Finally, both the sum and the upper envelope of a finite number of subparabolic functions are subparabolic.

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If E is any set in the xt-plane, we let E_c denote the part of E which is contained in the half plane $t \leq c$. Subparabolic functions satisfy the following important maximum principle. (Compare [13, p. 6].)

THEOREM 1. *Suppose that* w *is subparabolic in a domain* D *with boundary* Γ. *If, for some* c,

$$\limsup_{P \to Q,\ P \in D_c} w(P) \leq A, \tag{1.5}$$

for all $Q \in \Gamma_c$, *then* $w \leq A$ *in* D_c.

When D_c is unbounded, we must assume that Γ_c contains the point at infinity.

Proof. Let B be the least upper bound for w over D_c. If w assumes this value at $P_0 = (x_0, t_0) \in D_c$ and if $R = R(x_1, x_2; t_1, t_2)$ is any open rectangle containing P_0 such that $R \cup S \subset D$, then (1.2), (1.3), and (1.4) imply that $w(Q) = B$ for $Q = (\xi, \tau) \in S$, $t_1 \leq \tau \leq t_0$. In particular, $w(x_2, t_0) = B$; and proceeding by induction, we can find an increasing sequence $\{x_{2n}\}$ such that $P_n = (x_{2n}, t_0)$ approaches a point $Q_0 \in \Gamma_c$ and such that $w(P_n) = B$ for all n. From (1.5) we conclude that $B \leq A$. If u does not assume its least upper bound in D_c, we can find a sequence of points $\{P_n\}$ in D_c which approach $Q_0 \in \Gamma_c$ and such that $u(P_n) \to B$. Again we obtain $B \leq A$ from (1.5).

With Theorem 1 it is not difficult to obtain the following alternative characterization for subparabolic functions. (Compare [1, p. 194].)

COROLLARY 1. *A continuous function* w *is subparabolic in* D *if and only if, for each* u *parabolic in a subdomain* $\triangle$ *of* D, *the function* $w - u$ *satisfies the maximum principle of Theorem* 1 *in* $\triangle$.

In the sequel we shall require the following elementary lemma concerning the Poisson kernel

$$k(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

LEMMA 1. *Suppose that* $0 < \alpha < 1$, $\beta > 0$, $\gamma > 0$. *Then*

$$|k_x(x, \tau)| \leq C_1 k(x, t) \qquad (C_1 = C_1(\alpha, \beta, \gamma) < \infty), \tag{1.6}$$

for $0 < \tau < \alpha t$, $\beta > t$, $\gamma \leq |x|$. *Suppose that* $0 < \alpha < \beta < 1$, $\gamma > 0$. *Then*

$$k(x - y, \tau) \leq C_2 k(x, t) \qquad (C_2 = C_2(\alpha, \beta, \gamma) < \infty), \tag{1.7}$$

for $\alpha t \leq \tau \leq \beta t$, $y^2 \leq \gamma t$, $t > 0$.

Proof. The inequality (1.6) follows from a routine calculation. The restrictions $\tau \leq \beta t$, $y^2 \leq \gamma t$ imply that

$$-\frac{(x - y)^2}{4\tau} + \frac{x^2}{4t} \leq \frac{y^2}{4t(1 - \beta)} \leq \frac{\gamma}{4(1 - \beta)},$$

and we obtain inequality (1.7) with

$$C_2 = \frac{1}{\sqrt{\alpha}} \exp \frac{\gamma}{4(1 - \beta)}.$$

2. UNIQUENESS THEOREMS

We consider here some extensions, of Phragmén-Lindelöf type, of Theorem 1 for the case where D is the strip $0 < t < c$. We adopt the notation $w^+ = \text{Max}(w, 0)$.

THEOREM 2. *Suppose that* w *is subparabolic in the strip* $0 \leq t < c$, *that*

$$\int_{-\infty}^{\infty} \int_0^a k(x, b - t)\, w^+(x, t)\, dt\, dx < \infty \tag{2.1}$$

for all $0 < a < b < c$, *and that* $w(x, 0) \leq A$ *for all* x. *Then* $w \leq A$ *in this strip.*

By the first hypothesis we mean that w is subparabolic in some domain containing the strip $0 \leq t < c$.

Proof. If we replace w by the function $w - A$, then (2.1) still holds. Hence we can assume, without loss of generality, that $A = 0$. We begin by showing that

$$w(0, t) \leq 0 \tag{2.2}$$

for $0 < t < c$. For this, fix $0 < t < c$, $y > 0$, and consider the rectangle $R = R(-y, y; 0, c)$. Now $w \leq w^+$. With (1.2) and (1.4) we have

$$w(0, t) \leq \int_S G(0, t; \xi, \tau)\, w^+(\xi, \tau)\, dS$$

and, since $w(x, 0) \leq 0$ for all x, we obtain

$$w(0, t) \leq \int_T G(0, t; \xi, \tau)\, w^+(\xi, \tau)\, dT, \tag{2.3}$$

where T consists of that part of S not contained in the line $t = 0$. For the rectangle R, the kernel G is well known [3, p. 177], and if we let

$$h(y, t) = -2 \sum_{n=-\infty}^{\infty} k_x\{(4n + 1)y, t\},$$

then we can rewrite (2.3) as

$$w(0, t) \leq \int_0^t h(y, t - \tau)\, w^+(y, \tau)\, d\tau - \int_0^t h(-y, t - \tau)\, w^+(-y, \tau)\, d\tau. \tag{2.4}$$

With $|y| \geq \gamma = \sqrt{2t}$, it is not difficult to show that

$$|h(y, t)| \leq 2 \sum_{n=1}^{\infty} |k_x(ny, t)| \leq 4\, |k_x(y, t)|,$$

and (2.4) yields

$$(2.5)\qquad w(0,\,t) \le 4\int_0^t |k_x(y,\,t-\tau)|\,\{w^+(y,\,\tau) + w^+(-y,\,\tau)\}\,d\tau.$$

Fix b so that $t < b < c$. Then $0 < \tau < t$ implies that $0 < t - \tau < \alpha(b - \tau)$ and $b - \tau < \beta$, where $0 < \alpha < 1$, $\beta > 0$, and (2.5) and (1.6) yield

$$(2.6)\qquad w(0,\,t) \le 4C_1\int_0^t k(y,\,b-\tau)\{w^+(y,\,\tau) + w^+(-y,\,\tau)\}\,d\tau$$

for $\gamma \le y$. Integrating both sides of (2.6) over the interval $\gamma \le y \le \delta$, we conclude that

$$(\delta - \gamma)\,w(0,\,t) \le 4C_1\int_\gamma^\delta\int_0^t k(y,\,b-\tau)\{w^+(y,\,\tau) + w^+(-y,\,\tau)\}\,d\tau dy\,,$$

and hence that

$$(2.7)\qquad w(0,\,t) \le \frac{4C_1}{\delta - \gamma}\int_{-\infty}^{\infty}\int_0^t k(y,\,b-\tau)\,w^+(y,\,\tau)\,d\tau dy\,.$$

We obtain (2.2) from (2.1) and (2.7) by letting $\delta \to \infty$.

To complete the proof for Theorem 1, fix y and choose $0 < a_1 < b_1 < b < c$. If $0 < t < a_1$, then

$$\alpha(b - t) \le b_1 - t \le \beta(b - t), \qquad y^2 \le \gamma(b - t)\,,$$

where $0 < \alpha < \beta < 1$, $\gamma > 0$; applying (1.7) to (2.1), we conclude that

$$\int_{-\infty}^{\infty}\int_0^{a_1} k(x - y,\,b_1 - t)\,w^+(x,\,t)\,dt dx < \infty\,.$$

By the previous argument we obtain $w(y,\,t) \le 0$ for $0 < t < c$, and the proof is complete.

THEOREM 3. *Suppose that* w *is subparabolic in the strip* $0 < t < c$, *that* (2.1) *holds for all* $0 < a < b < c$, *and that*

$$\limsup_{\substack{(x,t) \to (x_0,0)\\ 0<t<c}} w(x,\,t) \le A\,,$$

for all x_0. *Then* $w \le A$ *in this strip.*

Proof. Again we can assume that $A = 0$. Define R, S and T as in the proof for Theorem 2 and, for $P \in R$, let

$$u(P) = \int_S G(P,\,Q)\,f(Q)\,dS\,,$$

where f is the continuous function equal to w^+ on T and equal to 0 on S — T. Now w - u is subparabolic in R,

$$\lim_{P \to Q,\ P \in R} \sup \{w(P) - u(P)\} \leq 0$$

for all $Q \in S$ [6, p. 369], and with Theorem 1 we conclude that $w \leq u$ in R. Hence we obtain (2.3), and the proof is completed as before.

The following results are immediate consequences of Theorem 3. (For a related theorem, see [2].)

COROLLARY 2. *Suppose that* u *is parabolic in the strip* $0 < t < c$, *that*

$$\int_{-\infty}^{\infty} \int_0^a k(x, b - t)\, |u(x, t)|\, dt\, dx < \infty$$

for all $0 < a < b < c$, *and that*

$$\lim_{\substack{(x,t) \to (x_0,0) \\ 0<t<c}} u(x, t) = A$$

for all x_0. *Then* $u \equiv A$ *in this strip.*

COROLLARY 3. *Suppose that* w *is subparabolic in the strip* $0 < t < c$, *that*

$$\sup_{0<t<b} \int_{-\infty}^{\infty} k(x, b - t)\, w^+(x, t)\, dx < \infty$$

for all $0 < b < c$, *and that*

$$\limsup_{\substack{(x,t) \to (x_0,0) \\ 0<t<c}} w(x, t) \leq A,$$

for all x_0. *Then* $w \leq A$ *in this strip.*

Corollary 2 is an extension of a well-known uniqueness theorem due to Tychonoff [14]. A subparabolic form for the Tychonoff theorem is as follows.

COROLLARY 4. *Suppose that* w *is subparabolic in the strip* $0 < t < c$, *that*

$$w(x, t) \leq Me^{ax^2} \qquad (0 < M < \infty,\ 0 < a < \infty)$$

in $0 < t < c$, *and that*

$$\limsup_{\substack{(x,t) \to (x_0,0) \\ 0<t<c}} w(x, t) \leq A$$

for all x_0. *Then* $w \leq A$ *in this strip.*

Proof. Fix $c_1 = \mathrm{Min}(1/8a, c)$. If $0 < b < c_1$, then

$$\int_{-\infty}^{\infty} k(x, b - t)\, w^+(x, t)\, dx \leq N \int_{-\infty}^{\infty} k\{x, 2(b - t)\}\, dx = N$$

for $0 < t < b$, $N = \sqrt{2}M$, and we conclude, from Corollary 3, that $w \leq A$ in $0 < t < c_1$. If $c_1 < c$, the proof is completed by a familiar step-by-step argument [14].

3. PARABOLIC MAJORANTS

We say that u is a majorant for w in a domain D if $w \leq u$ in D. We obtain here a necessary and sufficient condition for a function w, subparabolic in $0 < t < c$, to have a nonnegative parabolic majorant in this strip.

We begin with the following parabolic analogue of a result due to Littlewood [8, p. 193].

THEOREM 4. *Suppose that* w *is subparabolic in the strip* $0 < t < c$, *and that*

$$\sup_{0<t<b} \int_{-\infty}^{\infty} k(x, b - t)\, w^+(x, t)\, dx < \infty \tag{3.1}$$

for all $0 < b < c$. *Then for* $0 < a < c$ *the function*

$$u_a = u_a(x, t) = \int_{-\infty}^{\infty} k(x - y, t - a)\, w^+(y, a)\, dy \tag{3.2}$$

is nonnegative and parabolic in the strip $a < t < c$, *and*

$$w^+(x, t) \leq u_a(x, t) \tag{3.3}$$

there. Furthermore, for each fixed (x, t) *in the strip* $0 < t < c$, $u_a(x, t)$ *is nonincreasing as a function of* a *in the interval* $0 < a < t$.

Proof. Condition (3.1) implies that the integral in (3.2) is convergent in $a < t < c$, and hence that u_a is parabolic in this strip [15, p. 88]. For (3.3) let

$$w_1(x, t) = w^+(x, t + a) - u_a(x, t + a).$$

Then w_1 is subparabolic in $0 < t < c - a$,

$$\lim_{\substack{(x,t) \to (x_0,0) \\ 0<t<c-a}} w_1(x, t) = 0$$

for all x_0 [15, p. 89], and

$$\int_{-\infty}^{\infty} k(x, b - t)\, w_1^+(x, t)\, dx \leq \int_{-\infty}^{\infty} k(x, b - t)\, w^+(x, t + a)\, dx$$

for all $0 < b < c - a$. Appealing to (3.1) and Corollary 3, we conclude that $w_1 \leq 0$ in $0 < t < c - a$, and hence that (3.3) holds in $a < t < c$. For the last part, fix $0 < a_1 < a_2 < c$, and let

$$w_2(x, t) = u_{a_2}(x, t + a_2) - u_{a_1}(x, t + a_2).$$

Then w_2 is parabolic in $0 < t < c - a_2$,

$$\lim_{\substack{(x,t) \to (x_0, 0) \\ 0 < t < c - a_2}} w_2(x, t) = w^+(x_0, a_2) - u_{a_1}(x_0, a_2) \leq 0$$

for all x_0, and, by Fubini's theorem, if $0 < b < c - a_2$, then

$$\int_{-\infty}^{\infty} k(x, b - t)\, w_2^+(x, t)\, dt \leq \int_{-\infty}^{\infty} k(x, b - t)\, u_{a_2}(x, t + a_2)\, dx$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} k(x, b - t)\, k(x - y, t)\, dx \right\} w^+(y, a_2)\, dy$$

$$= \int_{-\infty}^{\infty} k(y, b)\, w^+(y, a_2)\, dy < \infty$$

for all $0 < t < b$. By Corollary 3, $w_2 \leq 0$ in $0 < t < c - a_2$, and we conclude that $u_{a_2} \leq u_{a_1}$ in the strip $a_2 < t < c$ as desired.

We require next the following analogue for the second theorem of Harnack.

THEOREM 5. *Suppose that* $\{u_n\}$ *is a nondecreasing sequence of functions which are parabolic in the strip* $0 < t < c$, *and for which*

$$\lim_{n \to \infty} u_n(0, b) < \infty$$

for all $0 < b < c$. *Then*

$$\lim_{n \to \infty} u_n(x, t) = u(x, t)$$

uniformly on each compact set in $0 < t < c$, *and* u *is parabolic in this strip.*

Proof. The uniform convergence follows from a result of Hadamard [5] and Pini [10, p. 429]. The fact that u is parabolic is a consequence of the parabolic analogue for the first theorem of Harnack [13, p. 13].

Combining the previous two theorems, we obtain the following result.

THEOREM 6. *Suppose that* w *is subparabolic in the strip* $0 < t < c$. *Then* w *has a nonnegative parabolic majorant in this strip if and only if*

$$\sup_{0<t<b} \int_{-\infty}^{\infty} k(x, b - t)\, w^{+}(x, t)\, dx < \infty \tag{3.4}$$

for all $0 < b < c$.

Proof. Suppose that w has such a majorant u. Then, by a well-known theorem due to Widder [15, p. 92], u has the representation

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t)\, d\mu(y)$$

in $0 < t < c$, where $\mu(y)$ is nondecreasing in y. If we fix $0 < b < c$ and apply Fubini's theorem, we obtain

$$\int_{-\infty}^{\infty} k(x, b - t)\, w^{+}(x, t)\, dx \leq \int_{-\infty}^{\infty} k(x, b - t)\, u(x, t)\, dx$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} k(x, b - t)\, k(x - y, t)\, dx \right\} d\mu(y) \tag{3.5}$$

$$= \int_{-\infty}^{\infty} k(y, b)\, d\mu(y) = u(0, b) < \infty$$

for $0 < t < b$. Hence (3.4) is necessary. To show that (3.4) is sufficient, define u_a as in (3.2) for $0 < a < c$. Then $w^{+} \leq u_a$ in $a < t < c$ for each fixed a, and $u_a(x, t)$ is nonincreasing in $0 < a < t$ for each fixed (x, t). By (3.4),

$$\lim_{a \to 0+} u_a(0, b) = \lim_{a \to 0+} \int_{-\infty}^{\infty} k(x, b - a)\, w^{+}(x, a)\, dx < \infty$$

for all $0 < b < c$, and we can apply Theorem 5 to conclude that

$$\lim_{a \to 0+} u_a(x, t) = u(x, t)$$

uniformly on each compact set in $0 < t < c$. Clearly, u is the desired majorant for w.

4. REPRESENTATION THEOREMS

With the aid of Theorem 6, we obtain a number of representation theorems for functions parabolic in the strip $0 < t < c$.

THEOREM 7. *A necessary and sufficient condition that* u *can be expressed as the difference of two functions which are nonnegative and parabolic in* $0 < t < c$ *is that* u *be parabolic in this strip and that*

$$\sup_{0<t<b} \int_{-\infty}^{\infty} k(x, b - t)|u(x, t)|\,dx < \infty \tag{4.1}$$

for all $0 < b < c$.

Proof. Suppose first that $u = v_1 - v_2$, where v_1 and v_2 are nonnegative and parabolic in $0 < t < c$. Then u is parabolic, $w = |u|$ has $v = v_1 + v_2$ as a nonnegative parabolic majorant in this strip, and condition (4.1) follows from Theorem 6. Conversely, if u is parabolic and if (4.1) holds, then, by Theorem 6, $w = |u|$ has a nonnegative parabolic majorant, say v_1, in $0 < t < c$. Since $v_2 = v_1 - u$ is nonnegative and parabolic in this strip, u has the desired representation.

Combining Theorem 7 with the Widder representation theorem [15, p. 92], we obtain the following result.

THEOREM 8. *A necessary and sufficient condition that* u *have the representation*

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t)\,d\mu(y) \tag{4.2}$$

in $0 < t < c$, *where the integral is absolutely convergent, is that* u *be parabolic in this strip and that* (4.1) *hold for all* $0 < b < c$.

Theorem 8 is equivalent to Theorem 2 of [4]. On the other hand, Theorem 8 is in slight disagreement with the following one-dimensional form of Theorem 4 of [11].

THEOREM. *A necessary and sufficient condition that* u *have the representation* (4.2) *in* $0 < t < c$ *is that* u *be parabolic in this strip and that, for some* $0 < M \leq 1/4c$,

$$\sup_{0<t<c} \int_{-\infty}^{\infty} |u(x, t)|\, e^{-Mx^2} dx < \infty. \tag{4.3}$$

If u is parabolic in $0 < t < c$ and if (4.3) holds, then (4.1) holds for all $0 < b < c$, and u has the desired representation, by Theorem 8. However, Widder has pointed out (oral communication) that (4.3) is not necessarily satisfied by functions with such a representation. For example let

$$u(x, t) = \frac{1}{\sqrt{4\pi(c - t)}} \exp \frac{x^2}{4(c - t)}$$

in $0 < t < c$. Then u is nonnegative and parabolic in this strip, u has the representation (4.2) by the Widder theorem, and (4.3) does not hold for any finite M.

Theorem 8 offers an alternative proof for the following theorem due to Rosenbloom and Widder [12].

THEOREM 9. *A necessary and sufficient condition that* u *have the representation* (4.2) *in* $0 < t < c$, *where*

$$\int_{-\infty}^{\infty} k(x, c)\,|d\mu(x)| \leq M \qquad (0 < M < \infty), \tag{4.4}$$

is that u *be parabolic in this strip and that*

$$\int_{-\infty}^{\infty} k(x, c - t)\,|u(x, t)|\,dx \le M \tag{4.5}$$

for $0 < t < c$.

Proof. If u has such a representation, then u is parabolic, and (4.5) can be obtained from (4.2), (4.4) and the Fubini theorem, as in (3.5). Conversely, suppose that u is parabolic and that (4.5) holds. Fix $0 < b < c$ and let $\triangle$ be any closed disk in $0 < t < c$ with center at (0, b). Then

$$k(x, b - t) \le C_3 k(x, c - t) \qquad (C_3 = C_3(\triangle) < \infty)$$

for $0 < t < b$ and $(x, t) \notin \triangle$, and we conclude from (4.5) that

$$\int_{-\infty}^{\infty} k(x, b - t)\,|u(x, t)|\,dx \le C_3 M + N < \infty$$

for $0 < t < b$, where N is an upper bound for $|u|$ in $\triangle$. Hence (4.1) holds for all $0 < b < c$, and u has the representation (4.2) by Theorem 8. (If (4.1) holds for a certain value of $b > 0$, it holds for all smaller values of $b > 0$.) For (4.4) we may assume that μ is normalized, in other words, that $\mu(x) = [\mu(x + 0) + \mu(x - 0)]/2$ for all x. Then

$$\mu(y_2) - \mu(y_1) = \lim_{t \to 0+} \int_{y_1}^{y_2} u(x, t)\,dx \tag{4.6}$$

for all $y_1 < y_2$ (see [16, p. 293] or [4]). Now (4.5) and (4.6) imply that

$$\int_{y_1}^{y_2} |d\mu(x)| \le \liminf_{t \to 0+} \int_{y_1}^{y_2} |u(x, t)|\,dx < \infty$$

for all $y_1 < y_2$, and we conclude, with the Helly theorem and the Helly-Bray theorem, that

$$\int_{y_1}^{y_2} k(x, c)\,|d\mu(x)| \le \liminf_{t \to 0+} \int_{y_1}^{y_2} k(x, c - t)\,|u(x, t)|\,dx \le M$$

for all $y_1 < y_2$. The proof is completed by letting $y_1 \to -\infty$ and $y_2 \to \infty$.

In conclusion we consider the following variant of Theorem 9.

THEOREM 10. *A necessary and sufficient condition that* u *have the representation*

$$u(x, t) = \int_{-\infty}^{\infty} k(x - y, t)\,f(y)\,dy \tag{4.7}$$

in $0 < t < c$, *where, for some* $p > 1$,

$$\int_{-\infty}^{\infty} k(x, c)\, |f(x)|^p\, dx \le M^p \qquad (0 < M < \infty), \tag{4.8}$$

is that u *be parabolic in this strip and that*

$$\int_{-\infty}^{\infty} k(x, c - t)\, |u(x, t)|^p\, dx \le M^p \tag{4.9}$$

for $0 < t < c$.

Proof. If u has such a representation, then u is parabolic and, with (4.7), (4.8), Jensen's inequality, and Fubini's theorem, we conclude that

$$\int_{-\infty}^{\infty} k(x, c - t)\, |u(x, t)|^p\, dx \le \int_{-\infty}^{\infty} k(x, c - t) \left\{ \int_{-\infty}^{\infty} k(x - y, t)\, |f(y)|^p\, dy \right\} dx$$

$$= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} k(x, c - t)\, k(x - y, t)\, dx \right\} |f(y)|^p\, dy$$

$$= \int_{-\infty}^{\infty} k(y, c)\, |f(y)|^p\, dy \le M^p$$

for $0 < t < c$, as desired. Conversely, suppose that u is parabolic and that (4.9) holds. Since (4.9) implies (4.5), u has the representation (4.2), and (4.6) holds for all $y_1 < y_2$. Now (4.9) also implies that the integrals

$$\int_{y_1}^{y_2} |u(x, t)|^p\, dx$$

are bounded for small t. Hence, by a theorem due to de La Vallée Poussin [7, p. 452], μ is absolutely continuous over each finite interval, and we obtain (4.7) with $f(x) = \mu'(x)$. Since

$$\lim_{t \to 0+} u(x, t) = f(x)$$

for almost all x [11, p. 194], (4.9) and Fatou's lemma yield

$$\int_{-\infty}^{\infty} k(x, c)\, |f(x)|^p\, dx \le \liminf_{t \to 0+} \int_{-\infty}^{\infty} k(x, c - t)\, |u(x, t)|^p\, dx \le M^p,$$

and the proof is complete.

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