THE ASYMMETRY OF CERTAIN ALGEBRAS OF FOURIER-STIELTJES TRANSFORMS

Edwin Hewitt

1. INTRODUCTION

Throughout the present paper, G will denote a locally compact Abelian group, and X its character group. We write the group operation as multiplication except in dealing with certain classical cases: no confusion should arise. (For all group-theoretic facts and terms not explained here, see [10].) The symbol R denotes the additive group of real numbers; T the multiplicative group of complex numbers of absolute value 1; N the additive group of all integers; Z(m) the additive group of integers modulo m $(m = 2, 3, \cdots)$; and Δ_p the additive group of p-adic integers $(p = 2, 3, 5, 7, 11, \cdots)$. For A and B in G, the symbol AB denotes the set $\{ab: a \in A,\ b \in B\}$.

Let $\mathfrak{B}$ (the Borel sets in G) be the smallest σ-algebra of subsets of G containing all compact sets. (For all set- and measure-theoretic terms and facts not explained here, see [3].) Let $\mathcal{M}(G)$ denote the set of all regular, countably additive, complex-valued, bounded Borel measures on G. For $\lambda \in \mathcal{M}(G)$, one can write

$$\lambda = \lambda_1 - \lambda_2 + i(\lambda_3 - \lambda_4), \tag{1.1}$$

where λ_1, λ_2, λ_3, and λ_4 are nonnegative real measures in $\mathcal{M}(G)$, λ_1 is singular with respect to λ_2, and λ_3 is singular with respect to λ_4. Let $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$. We say that λ is concentrated on a set $E \in \mathfrak{B}$ if $|\lambda|(E') = 0$.

Let $\mathfrak{C}_\infty(G)$ denote the set of all continuous complex-valued functions on G each of which is arbitrarily small in absolute value outside of some compact set. It is well known that $\mathcal{M}(G)$ yields a concrete representation of the conjugate space of $\mathfrak{C}_\infty(G)$ (under the uniform norm in $\mathfrak{C}_\infty(G)$), the mapping

$$f \to \int_G f(x)\, d\lambda(x) \qquad (\lambda \in \mathcal{M}(G))$$

being the general bounded linear functional on $\mathfrak{C}_\infty(G)$. When each λ in $\mathcal{M}(G)$ is given its norm as a linear functional, $\mathcal{M}(G)$ becomes a complex Banach space.

It is also well known that $\mathcal{M}(G)$ is a Banach algebra under the operation of convolution:

$$\lambda * \mu(f) = \int_G \int_G f(xy)\, d\mu(y)\, d\lambda(x) \tag{1.2}$$

for λ, $\mu \in \mathcal{M}(G)$ and $f \in \mathfrak{C}_\infty(G)$ (see for example [8], 1.4.6). The value of the measure $\lambda * \mu$ for the Borel set E is

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(1.3) $$\int_G \mu(x^{-1}E)\, d\lambda(x)\,.$$

For $\lambda \in \mathcal{M}(G)$, let $\tilde{\lambda}$ be the element of $\mathcal{M}(G)$ such that

(1.4) $$\tilde{\lambda}(f) = \overline{\int_G \overline{f(x^{-1})}\, d\lambda(x)}$$

for all $f \in \mathfrak{C}_\infty(G)$. It is clear that

(1.5) $$\tilde{\lambda}(E) = \overline{\lambda(E^{-1})}$$

for all $E \in \mathfrak{B}$. Also, we have

(1.6) $$\int_G \chi(x)\, d\tilde{\lambda}(x) = \overline{\int_G \chi(x)\, d\lambda(x)}$$

for all $\chi \in X$ and $\lambda \in \mathcal{M}(G)$.

Now suppose that $\mathcal{M}(G)$ admits an adjoint operation $\lambda \to \lambda^*$ under which it is symmetric in the sense of Gel'fand, Raĭkov, and Šilov ([2], p. 139). The uniqueness theorem for Fourier-Stieltjes transforms and (1.6) show that λ^* must be equal to $\tilde{\lambda}$ for all $\lambda \in M(G)$.

Šreĭder ([13], pp. 311-313) has shown that there is a measure $\sigma \in \mathcal{M}(R)$ and a multiplicative linear functional M_0 on $\mathcal{M}(R)$ such that $M_0(\sigma) = 1$ and $M_0(\tilde{\sigma}) = 0$; that is, $\mathcal{M}(R)$ is asymmetric. We shall extend Šreĭder's result to a large class of locally compact Abelian groups.

1.1. MAIN THEOREM. *Let* G *be a locally compact Abelian group such that every neighborhood of the identity contains an element of infinite order.*† *Then there exist a measure* $\sigma \in \mathcal{M}(G)$ *and a multiplicative linear functional* M_0 *on* $\mathcal{M}(G)$ *such that* $M_0(\sigma) = 1$ *and* $M_0(\tilde{\sigma}) = 0$. *Thus* $\mathcal{M}(G)$ *is asymmetric.*

Theorem 1.1 can be rephrased as follows.

1.2. *Let* X *be the character group of a group* G *such that every neighborhood of the identity in* G *contains an element of infinite order. Then the algebra of all functions on* X *that are Fourier-Stieltjes transforms is asymmetric.*

In Sections 2 to 5, we carry out the proof of Theorem 1.1. In Section 6, we draw some inferences from it.

2. THE FIRST STEP

This section is modelled on Šreĭder [13].

2.1. THEOREM. *Suppose that* G *contains a homeomorphic image* P *of Cantor's ternary set containing a countable subset* N_0 *such that the elements of* $P \cap N_0' = Q$

† The writer is indebted to Professor Deane Montgomery for this simple characterization of the class of groups studied and for several helpful conversations. Professor Walter Rudin and Dr. John H. Williamson have also made useful comments.

are independent in G. *That is, if* $x_1, \cdots, x_s \in Q$ *and* $\alpha_1, \cdots, \alpha_s$ *are integers, the equality*

$$x_1^{\alpha_1} \cdots x_s^{\alpha_s} = e \tag{2.1}$$

implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_s = 0$. *Then Theorem* 1.1 *holds for the group* G.

We proceed to prove Theorem 2.1.

2.2. LEMMA. *For every positive integer* n *and every* $t \in G$, *the set* $(Q^n t) \cap (Q^{-1})$ *contains no more than* n + 1 *points.*

Proof. Assume the contrary. Then for some n, there exist n + 2 equalities

$$y^{(i)} x_1^{(i)} x_2^{(i)} \cdots x_n^{(i)} = t^{-1} \qquad (i = 1, 2, \cdots, n + 2), \tag{2.2}$$

where the elements $y^{(1)}, y^{(2)}, \cdots, n^{(n+2)}$ are all distinct. This implies that

$$y^{(1)} x_1^{(1)} x_2^{(1)} \cdots x_n^{(1)} = y^{(i)} x_1^{(i)} x_2^{(i)} \cdots x_n^{(i)} \qquad (i = 2, 3, \cdots, n + 2), \tag{2.3}$$

If all of the x's and y's belong to Q, then the independence condition on Q implies that there exist distinct indices $i_2, i_3, \cdots, i_{n+2}$ such that $y^{(j)} = x_{i_j}^{(2)}$ $(j = 2, 3, \cdots, n + 2)$. This contradiction proves the lemma.

2.3. LEMMA (see Raĭkov's construction in [2], pp. 184-186). *Let* $\mathfrak{F}$ *be a nonvoid family of* σ*-compact subsets of* G *with the following properties:*

$$\text{if } A \in \mathfrak{F},\ B \text{ is } \sigma\text{-compact, and } B \subset A, \text{ then } B \in \mathfrak{F}; \tag{2.4}$$

$$\{A_n\}_{n=1}^{\infty} \subset \mathfrak{F} \text{ implies } \bigcup_{n=1}^{\infty} A_n \in \mathfrak{F}; \tag{2.5}$$

$$A, B \in \mathfrak{F} \text{ implies } AB \in \mathfrak{F}; \tag{2.6}$$

$$A \in \mathfrak{F} \text{ and } t \in G \text{ imply } tA \in \mathfrak{F}. \tag{2.7}$$

Let $\mathscr{R}$ *be the set of all measures* $\mu \in \mathscr{M}(G)$ *such that* $|\mu|$ *is concentrated on some element of* $\mathfrak{F}$, *and let* $\mathscr{I}$ *be the set of all measures* $\mu \in \mathscr{M}(G)$ *such that* $|\mu|(A) = 0$ *for all* $A \in \mathfrak{F}$. *Then* $\mathscr{I}$ *is a closed ideal in* $\mathscr{M}(G)$, *and* $\mathscr{R}$ *is a closed subalgebra of* $\mathscr{M}(G)$. *Furthermore,* $\mathscr{M}(G)$ *is the direct sum of* $\mathscr{R}$ *and* $\mathscr{I}$: *every* $\mu \in \mathscr{M}(G)$ *can be written in just one way as* $\mu = \mu' + \mu''$, *where* $\mu' \in \mathscr{R}$ *and* $\mu'' \in \mathscr{I}$.

Proof. Consider first $\mathscr{I}$. It is obvious that $\mathscr{I}$ is a linear subspace of $\mathscr{M}(G)$. An elementary argument, which we omit, shows that $\mathscr{I}$ is closed. To show that $\mathscr{I}$ is an ideal in $\mathscr{M}(G)$, consider any $\mu \in \mathscr{I}$ and $\lambda \in \mathscr{M}(G)$. Write μ in the form (1.1). It is clear that $\mu_1, \mu_2, \mu_3, \mu_4$ are in $\mathscr{I}$. If now $A \in \mathfrak{F}$, we have $t^{-1} \in \mathfrak{F}$ for all $t \in G$, and hence $\mu_j(t^{-1}A) = 0$ $(j = 1, 2, 3, 4)$. Therefore we have, by (1.3),

$$\lambda * \mu_j(A) = \int_G \mu(t^{-1}A)\, d\lambda(t) = 0.$$

Therefore $\lambda * \mu \in \mathscr{I}$ if $\lambda \geq 0$ and $\mu \in \mathscr{I}$. Writing a general λ in the form (1.1), we infer that $\lambda * \mu \in \mathscr{I}$. That is, $\mathscr{I}$ is a closed ideal in $\mathscr{M}(G)$.

Next consider $\mathscr{R}$. As above, it is easy to see that $\mathscr{R}$ is a closed linear subspace of $\mathscr{M}(G)$. Suppose that $\lambda, \mu \in \mathscr{R}$, and that $\lambda, \mu \geq 0$. Then λ and μ are both concentrated on a certain set $A \in \mathfrak{F}$. Let $B = \bigcup_{n=1}^{\infty} A^n$. Then $B \in \mathfrak{F}$, λ and μ are concentrated on B, and $B^2 \subset B$. Now, for every $E \in \mathfrak{B}$, we have

$$\lambda * \mu(E) = \int_B \lambda(t^{-1}E)\, d\mu(t) = \int_B \lambda(t^{-1}(E \cap B))\, d\mu(t) + \int_B \lambda(t^{-1}(E \cap B'))\, d\mu(t).$$

It is easy to see that $B^{-1}(E \cap B') \subset B'$, so that the last integral on the right vanishes for all $E \in \mathfrak{B}$. Therefore $\lambda * \mu$ is concentrated on B. For general λ, $\mu \in \mathscr{R}$, write λ and μ as linear combinations of nonnegative measures as in (1.1). Each summand λ_j and μ_k is in $\mathscr{R}$, hence $\lambda_j * \mu_k$ is in $\mathscr{R}$, and finally $\lambda * \mu$ is in $\mathscr{R}$.

It remains to show that every μ can be written in one and only one way as $\mu' + \mu''$, where $\mu' \in \mathscr{R}$ and $\mu'' \in \mathscr{I}$. As before, we may suppose that $\mu \geq 0$. For $E \in \mathfrak{B}$, let

$$\mu'(E) = \sup\{\mu(B): B \in \mathfrak{F},\ B \subset E\},$$

$$\mu''(E) = \mu(E) - \mu'(E).$$

Then $\mu' \in \mathscr{R}$ and $\mu'' \in \mathscr{I}$. Uniqueness is also easy to establish. We omit the details. This completes the proof of Lemma 2.3.

2.4. LEMMA. *Let* $\mathfrak{F}$, $\mathscr{I}$, *and* $\mathscr{R}$ *be as in Lemma* 2.3. *Then the mapping*

$$\mu \to \mu'(G) \tag{2.8}$$

is a multiplicative linear functional on $\mathscr{M}(G)$.

Proof. The mapping (2.8) is clearly a linear functional. Since $\mathscr{I}$ is an ideal, we see that $(\lambda * \mu)' = \lambda' * \mu'$ for all λ, $\mu \in \mathscr{M}(G)$. Since $\lambda' * \mu'(G) = \lambda'(G) \cdot \mu'(G)$, the lemma follows.

2.5. *Proof of Theorem* 2.1. Consider the smallest family $\mathfrak{F}$ of sets that contains P and satisfies conditions (2.4) to (2.7). Since P is a homeomorph of Cantor's ternary set, there is a nonnegative, countably additive Borel measure σ on P such that $\sigma(P) = 1$ and $\sigma(\{x\}) = 0$ for all points $x \in P$. We extend σ to all Borel sets E in G: $\sigma(E) = \sigma(E \cap P)$. Clearly σ is concentrated on Q, since σ vanishes for points. Since $Q \in \mathfrak{F}$, we have $\sigma' = \sigma$, for this choice of $\mathfrak{F}$. Since $\mathfrak{F}$ is the least family of sets containing P and satisfying the conditions (2.4) to (2.7), Lemma (2.2) implies that the set Q^{-1} intersects every set in $\mathfrak{F}$ only in a countable set. The measure $\tilde{\sigma}$ is concentrated on Q^{-1}, and hence by the definition of the ideal $\mathscr{I}$, we have $\tilde{\sigma} \in \mathscr{I}$, that is, $\tilde{\sigma}' = 0$. Therefore the multiplicative linear functional M_0 defined by (2.8) carries $\tilde{\sigma}$ into $\sigma(Q) = 1$ and carries σ into 0. Thus $\mathscr{M}(G)$ is asymmetric.

3. THE SECOND STEP

We here prove a structure theorem for locally compact Abelian groups of no great novelty, but perhaps of some interest on its own account, and in any case essential for the proof of Theorem 1.1.

3.1. THEOREM. *Let* G *be a locally compact Abelian group such that every neighborhood of the identity contains an element of infinite order. Then* G *contains a subgroup homeomorphic and algebraically isomorphic to one of the following:*

(3.1) *the group* R *with a topology no stronger than its usual topology;*

(3.2) *the group* T *with its usual topology;*

(3.3) *a full direct product*

$$\mathop{P}_{k=1}^{\infty} Z(p_k^{n_k}),$$

where $\{p_k\}_{k=1}^{\infty}$ *is a sequence of strictly increasing prime numbers and* $\{n_k\}_{k=1}^{\infty}$ *is a sequence of positive integers;*

(3.4) *the* p-*adic integers* Δ_p *for some prime* p.

Proof. Let U be an open neighborhood of e in G such that $U^{-1} = U$ and $\bar{U}$ is compact. Let H_1 be the open and closed subgroup $\bigcup_{n=1}^{\infty} U^n$. A standard theorem in the theory of locally compact Abelian groups ([12], p. 274, Theorem 5) asserts that H_1 is a direct product:

$$H_1 = R^a \times N^b \times H_2, \tag{3.5}$$

where a, b are nonnegative integers and H_2 is a compact group. If a is positive, we have (3.1). If a is zero, then H_2 is a compact open subgroup of G and must contain an element of infinite order, say x. Let H_3 be the closure of the subgroup $\{x^n\}_{n=-\infty}^{\infty}$. Let H_4 be the connected component of the identity in H_3. Suppose that $H_4 \neq \{e\}$. Then H_4 is an infinite, compact, connected group with a countable dense subset and hence has an open basis of cardinal number less than or equal to $2^{\aleph_0}$. Another well-known theorem ([12], p. 268, Example 67) asserts that there is a continuous homomorphism ϕ of R into H_4. The homomorphism ϕ can be made one-to-one if the character group of H_4 contains two independent elements, and the image of R under ϕ will be the group T otherwise. Thus we have (3.1) or (3.2) if $H_4 \neq \{e\}$.

Suppose finally that $H_4 = \{e\}$. Then H_3 is an infinite, totally disconnected, compact monothetic group [4], and its character group Y is a subgroup of the multiplicative group $\{e^{2\pi i r}\}$, where r runs through all rational numbers. Thus Y is an algebraic direct product of primary groups ([9], p. 5, Theorem 1). Let S_p be the subgroup of Y consisting of the elements whose order is a power of the prime p. Then $Y = S_2 \otimes S_3 \otimes S_5 \otimes \cdots$. A finite group S_p containing an element distinct from the identity is clearly isomorphic to $Z(p^n)$ for some positive integer n. If all of the S_p's are finite, then H_3 is the full direct product of the character groups of the S_p's. Each of these being isomorphic to S_p, we infer that in this case H_3 has the form (3.3). An infinite group S_p must be the group of all numbers of the form

$$e^{2\pi i(s/p^n)} \qquad (n = 1, 2, 3, \cdots; \ s = 0, 1, 2, \cdots, p^n - 1).$$

That is, such an S_p is the p^∞ group. The character group of the p^∞ group is the group of p-adic integers, in their usual topology (see for example [7]). Thus H_3 in this case is a full direct product of compact groups one of which is the group of p-adic integers. Thus we are in case (3.4).

3.2 *Note.* Groups of the sorts described in (3.1) to (3.4) obviously contain arbitrarily small elements of infinite order. Hence the converse of Theorem 3.1 holds. Obviously too there exist compact infinite Abelian groups without arbitrarily small elements of infinite order.

4. CONSTRUCTION OF THE SET P

We now show that every group of the form (3.1), (3.2), (3.3), or (3.4) contains a subset P satisfying the conditions set forth in Theorem 2.1.

4.1. *The group* R. This case has been dealt with, albeit incompletely, by Šreĭder [13]. He cites a construction given by J. v. Neumann [11] of a perfect set S of algebraically independent real numbers. The set S consists of all numbers

$$a(t) = \sum_{k=0}^{\infty} \frac{2^{2^{[tk]}}}{2^{2^{k^2}}} \qquad (t > 0). \tag{4.1}$$

(For a real number u, we write [u] for the integral part of u.) v. Neumann shows that S consists solely of algebraically independent real numbers. The set S is clearly dense in itself, but is not closed. The function a defined on $0 < t < \infty$ by (4.1) is strictly increasing, right-continuous everywhere, and left-continuous exactly at irrational points. Hence the numbers $\sup\{a(t): t < r\} = b(r)$ (r rational and positive) comprise the set $\overline{S} \cap S'$. The set $\overline{S}$, being perfect, contains a homeomorph P of Cantor's ternary set ([6], pp. 318-323). Let $N_0 = \{b(r): r$ rational and positive, $b(r) \in P\}$. Then P and N_0 satisfy the hypothesis of Theorem 2.1. For, although the mapping of R into the group G may be only a continuous and not a bicontinuous isomorphism, still, on the compact set P, it is bicontinuous.

4.2. *The group* T. Let u be any fixed number a(t) $(t > 0)$ as in (4.1). Parametrize T as the group of all numbers

$$\{e^{2\pi i x/u}\}_{0 \leq x < u}. \tag{4.2}$$

Let

$$a_1(t) = e^{2\pi i a(t)/u} \qquad (t > 0). \tag{4.3}$$

Then the set $S_1 = \{a_1(t)\}_{t>0}$ has all of the properties ascribed to S in 4.1, and sets P and N_0 with the properties required in Theorem 2.1 can be constructed exactly as was done in 4.1.

4.3. *The group* $P_{k=1}^{\infty} Z(p_k^{n_k})$. In 4.3 alone, we denote by H the group $P_{k=1}^{\infty} Z(p_k^{n_k})$. We may think of H as the set of all sequences $a = \{a_1, a_2, a_3, \cdots\}$, where all of the a_k are integers $(0 \leq a_k < p_k^{n_k})$ and $(a+b)_k = a_k + b_k$, the addition being carried out modulo $p_k^{n_k}$ $(k = 1, 2, 3, \cdots)$. H has the topology of a countable Cartesian product of finite discrete spaces, and is hence metrizable, zero-dimensional, and compact. The construction of our set P in this case is suggested by v. Neumann's construction (4.1). For every real number t $(1 \leq t < 2)$, let A(t) be the sequence in H such that

$$A(t)_k = \begin{cases} 1 & \text{if } k = [2^{t+l}],\ l = 0, 1, 2, \cdots, \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

It is easy to see that the mapping $t \to A(t)$ $(1 \leq t < 2)$ is one-to-one. Let $S = \{A(t)\}_{1 \leq t < 2}$. It is simple to show that if $\{t_n\}_{n=1}^{\infty}$ is a decreasing sequence of numbers $(1 \leq t_n < 2)$ with limit u, or an increasing sequence with irrational limit u, then $\lim_{n\to\infty} A(t_n) = A(u)$. Hence the only limit points of S not in S are elements of the form $\lim_{n\to\infty} A(t_n)$, where $\{t_n\}_{n=1}^{\infty}$ is a strictly increasing sequence with rational limit u. It is easy to show that there is just one such limit point for every rational u $(1 < u \leq 2)$, and hence $\bar{S} \cap S'$ is a countable set. It is clear that S is dense in itself. The set $\bar{S}$, as a perfect subset of the zero-dimensional compact metric space H, is homeomorphic to the Cantor ternary set ([1], p. 119, Satz VI).

It remains to show that the elements of S are independent in H. Suppose that $1 \leq t_1 < t_2 < \cdots < t_m < 2$ and that

(4.5) $$\alpha_1 A(t_1) + \cdots + \alpha_m A(t_m) = 0$$

in H, where $\alpha_1, \cdots, \alpha_m$ are nonzero integers. Choose the positive integer l so large that the following conditions hold. First,

(4.6) $$l > \max\left\{\frac{-\log(4 - 2^{t_m})}{\log 2}, \frac{-\log(2^{t_m} - 2^{t_{m-1}})}{\log 2}\right\}.$$

Second, if $k = [2^{t_{m+1}}]$, then the inequality

(4.7) $$p_n^{n_k} > |\alpha_m|$$

is to hold. If (4.6) holds, a routine calculation shows that the only entry at the kth place in $\Sigma_{i=1}^m \alpha_i A(t_i)$ is the number α_m. Hence we must have $\alpha_m \equiv 0 \pmod{p_k^{n_k}}$, and by (4.7), we must have $\alpha_m = 0$. This contradiction shows that S consists solely of independent elements. Thus the set $\bar{S}$ and the set $\bar{S} \cap S'$ can be used as the sets P and N_0, respectively, in Theorem 2.1.

4.4. *The group* Δ_p. This group can be thought of as the set of all sequences $a = \{a_1, a_2, \cdots\}$, where all the a_k are integers $(0 \leq a_k \leq p)$, and where addition is defined by induction (see for example [5], pp. 106-109). Write $a_1 + b_1 = d_1 p + r_1$, where $0 < r_1 < p$ and $d = 0$ or $d = 1$. Then $(a + b)_1 = r_1$. If $r_1, r_2, \cdots, r_k$ and $d_1, d_2, \cdots, d_k$ have already been defined, write $d_k + a_{k+1} + b_{k+1} = d_{k+1}p + r_{k+1}$, where $0 \leq r_{k+1} < p$ and d_{k+1} is a nonnegative integer. Then $(a + b)_{k+1} = r_{k+1}$. The group Δ_p has the topology of a Cartesian product of a countable number of finite discrete spaces and is hence metrizable, zero-dimensional, and compact. We define the set $S \subset \Delta_p$ as the set of all sequences A(t) $(1 \leq t < 2)$, just as in (4.4). The set $S \subset \Delta_p$ has precisely the same topological properties as the set $S \subset P_{k+1}^{\infty} Z(p_k^{n_k})$ that was discussed in 4.3. To show that $\bar{S}$ can be used as P, and $\bar{S} \cap S'$ as N_0, in Theorem 2.1, we need therefore only show that the elements of S are independent in Δ_p. Suppose that $1 \leq t_1 < t_2 < \cdots < t_m < 2$ and that

(4.8) $$\alpha_1 A(t_1) + \cdots + \alpha_m A(t_m) = 0 \quad \text{in } \Delta_p,$$

where $\alpha_1, \cdots, \alpha_m$ are nonzero integers. We may suppose that $\alpha_m > 0$. Consider a positive α_j $(j < m)$ and an index $k = [2^{n+t_j}]$ $(n = 0, 1, 2, \cdots)$. Suppose that

(4.9) $$\alpha_j = b_0^{(j)} + b_1^{(j)} p + b_2^{(j)} p^2 + \cdots + b_{v_j}^{(j)} p^{v_j},$$

where $0 \leq b_i < p$ for $i = 0, 1, \cdots, v_j$ (we shall also use the same representation for α_m). Then the entries in the sequence $\alpha_j A(t_j)$ at the places numbered

$$k, k + 1, \cdots, k + v_j$$

will be $b_0, b_1, \cdots, b_{v_j}$, respectively, if n is large enough. The entries in the places $k + v_j + 1, k + v_j + 2, \cdots, [2^{n+1+t_j}]$ (if any) will be zero. Now suppose that n is chosen so large that the inequalities

$$n > \frac{\log v_j - \log (2^{t_j} - 2^{t_j - 1})}{\log 2} \tag{4.10}$$

hold for all j with positive α_j. Suppose also that

$$n > - \frac{\log (2^{t_1 + 1} - 2^{t_m})}{\log 2} \tag{4.11}$$

and

$$n > - \frac{\log (2^{t_1} - 2^{t_m - 1})}{\log 2} . \tag{4.12}$$

Let $B = \Sigma \alpha_j A(t_j)$, summed over all j with positive α_j. Then B has zero entries at all of the places $[2^{n+t_m}], [2^{n+t_m}] + 1, \cdots, [2^{n+1+t_m}]$. Hence B can not affect the entries of $\alpha_m A(t_m)$ in this interval. Thus, if (4.8) is to hold, the entries

$$b_0^{(m)}, b_1^{(m)}, \cdots, b_{v_m}^{(m)}, 0, 0, \cdots, 0$$

of $\alpha_m A(t_m)$ in the places

$$[2^{n+t_m}], [2^{n+t_m}] + 1, \cdots, [2^{n+1+t_1}] \tag{4.13}$$

must be cancelled by the entries in the sequence $C = \Sigma \alpha_j A(t_j)$, where the sum is taken over all j with α_j negative. Suppose that there are s such α_j's. As before, if n is large enough, say $n > L_0$, every $\alpha_j A(t_j)$ will have the entry $p - 1$ in every place of the interval (4.13). For (4.8) to hold, a simple calculation shows that we must have

$$s(p - 1) + b_{v_m}^{(m)} + r = sp \tag{4.14}$$

(r is the "carried-over" number from the preceding place), again if n is sufficiently large, say $n > L_1$. Suppose that $\alpha_1 > 0$. If (4.14) and (4.8) hold, we have entries 0 in the sequence $\Sigma_{i=1}^{m} \alpha_i A(t_i)$ for all places

$$[2^{n+t_m}] + v_m, [2^{n+t_m}] + v_m + 1, \cdots, [2^{n+1+t_1}] + u - 1,$$

where u is defined by $b_0^{(1)} = \cdots = b_{u-1}^{(1)} = 0$, $0 < b_u^{(1)} < p$. If (4.8) holds, we must have

$$b_u^{(1)} + s(p - 1) + s \equiv 0 \pmod{p},$$

a contradiction. If $\alpha_1 < 0$, a similar contradiction arises. It is therefore impossible for (4.8) to hold, and this completes the proof that S consists of independent elements.

5. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is now complete. Suppose that every neighborhood of e in G contains an element of infinite order. Then by Theorem 3.1, G contains a subgroup of one of the forms (3.1) to (3.4). By Section 4, G contains a homeomorph P of Cantor's ternary set with a countable subset N_0 such that $P \cap N_0'$ consists solely of independent elements. By Theorem 2.1, $\mathcal{M}(G)$ admits an asymmetric multiplicative linear functional.

6. VARIOUS CONSEQUENCES OF THE MAIN THEOREM

We now draw some inferences from Theorem 1.1. Let $\mathfrak{S}$ denote the space of all multiplicative linear functionals on $\mathcal{M}(G)$, and write as usual $\hat{\mu}$ for the function on $\mathfrak{S}$ such that $\hat{\mu}(M) = M(\mu)$ for all $\mu \in \mathcal{M}(G)$ and $M \in \mathfrak{S}$. Let $\mathfrak{S}$ have the weakest topology under which all of the functions $\hat{\mu}$ are continuous. Let $\mathfrak{S}_0$ denote the set $\mathfrak{S}$, with the topology under which the sets $\{M: M \in \mathfrak{S}, M(\mu) \neq 0\}$ $(\mu \in \mathcal{M}(G))$ form a subbasis for open sets. For each $\chi \in X$, the functional P_χ, where

$$P_\chi(\mu) = \int_G \chi(x)\, d\mu(x),$$

is clearly an element of $\mathfrak{S}$. We can identify X with the subset $\{P_\chi\}_{\chi \in X}$ of $\mathfrak{S}$; in fact, the mapping $\chi \to P_\chi$ is a homeomorphism of X into $\mathfrak{S}$.

6.1. THEOREM. *Let* G *be as in Theorem* 1.1. *Then* X *is not dense in* $\mathfrak{S}$.

Proof. This follows from the fact that any M lying in the closure in $\mathfrak{S}$ of the set of functionals $\{P_\chi\}_{\chi \in X}$ must be symmetric.

6.2. THEOREM. *Let* G *be as in Theorem* 1.1. *Then there is a measure* $\lambda \in \mathcal{M}(G)$ *such that* $|\hat{\lambda}(P_\chi)| \geq 1$ *for all* $\chi \in X$ *and* λ *has no inverse in* $\mathcal{M}(G)$.

Proof. Let ε_e be the measure such that $\varepsilon_e(A) = 0$ or 1 according as $e \in A'$ or $e \in A$. Then ε_e is the unit of $\mathcal{M}(G)$, and $M(\varepsilon_e) = 1$ for all $M \in \mathfrak{S}$. Set $\lambda = \sigma - \tilde{\sigma} - e$, where σ is the measure referred to in Theorem 1.1. For M_0 as in Theorem 1.1, we plainly have $\hat{M_0}(\lambda) = 0$, so that λ has no inverse in M(G). For $\chi \in X$, we also have $|\hat{\lambda}(P_\chi)| = |1 + 2i\mathfrak{I}\hat{\sigma}(P_\chi)| \geq 1$.

6.3. *Note.* Theorem 6.2 is a generalization of a theorem stated by Wiener and Pitt ([14], p. 434, Theorem 3) for the case G = R. Their proof seems incomplete, however, and the proof given by Šreĭder for the case G = R ([13], p. 314, Theorem 6) is the first satisfactory one known to the writer.

6.4. THEOREM. *Let* G *be as in Theorem* 1.1. *Then the topological space* $\mathfrak{S}_0$ *has a strictly weaker topology than* $\mathfrak{S}$, *and* $\mathfrak{S}_0$ *fails to satisfy Hausdorff's separation axiom.*

Proof. The uniqueness theorem for Fourier-Stieltjes transforms implies that $\{P_\chi\}_{\chi \in X}$ is dense in $\mathfrak{S}_0$. Since this set is not dense in $\mathfrak{S}$ (Theorem 6.1), $\mathfrak{S}_0$ has a strictly weaker topology than $\mathfrak{S}$. If $\mathfrak{S}_0$ were a Hausdorff space, then the topologies on $\mathfrak{S}$ and $\mathfrak{S}_0$ would be identical.

6.5. THEOREM. *Let* G *be as in Theorem* 1.1. *Then there exists a measure* $\lambda \in \mathcal{M}(G)$ *such that the function* $|\hat{\lambda}|$ *on* $\mathfrak{S}$ *does not have the form* $\hat{\mu}$ *for any* $\mu \in \mathcal{M}(G)$.

Proof. Assume the contrary. Let $\lambda \in \mathcal{M}(G)$, let M_1 be a fixed element of $\mathfrak{S}$, and δ a positive real number. The function on $\mathfrak{S}$ whose value at $M \in \mathfrak{S}$ is

$$\max\{\delta - |\hat{\lambda}(M) - \hat{\lambda}(M_1|, 0\}$$

then has the form $\hat{\mu}$ for some $\mu \in \mathcal{M}(G)$. Clearly, $\hat{\mu}$ differs from zero only in the set $\{M: M \in \mathfrak{S}, |\mu(M) - \mu(M_1| < \delta\}$; hence the $\mathfrak{S}_0$-topology is the same as the $\mathfrak{S}$-topology. This contradicts Theorem 6.4.

6.6. *Note.* Theorem 6.5 does *not* assert that there is a $\lambda \in M(G)$ for which $|\hat{\lambda}|$, considered on $\{P_\chi\}_{\chi \in X}$ alone, fails to have the form $\hat{\mu}$. So far as the writer knows, the existence of such a λ is an open problem.

REFERENCES

1. P. S. Aleksandrov and H. Hopf, *Topologie I,* Grundlehren der Math. Wiss. in Einzeldarstellungen, Band XLV. Julius Springer, Berlin, 1935.
2. I. M. Gel'fand, D. A. Raĭkov, and G. E. Šilov, *Commutative normed rings*, Amer. Math. Soc. Translations (2) 5 (1957), 115-220. (Uspehi Mat. Nauk (N.S.) 1, no. 2(12) (1946), 48-146.)
3. P. R. Halmos, *Measure theory,* Van Nostrand, New York, 1950.
4. P. R. Halmos and H. Samelson, *On monothetic groups*, Proc. Nat. Acad. Sci. U. S. A. 28 (1942), 254-258.
5. H. Hasse, *Zahlentheorie,* Akademie-Verlag, Berlin, 1949.
6. F. Hausdorff, *Grundzüge der Mengenlehre,* von Veit, Leipzig, 1914.
7. E. Hewitt and H. S. Zuckerman, *A group-theoretic method in approximation theory,* Ann. of Math. (2) 52 (1950), 557-567.
8. ———, *Finite dimensional convolution algebras*, Acta Math. 93 (1955), 67-119.
9. I. Kaplansky, *Infinite Abelian groups*, University of Michigan Publications in Mathematics, no. 2. Univ. of Mich. Press, Ann Arbor, 1954.
10. L. H. Loomis, *An introduction to abstract harmonic analysis,* Van Nostrand, New York, 1953.
11. J. v. Neumann, *Ein System algebraisch unabhängiger Zahlen,* Math. Ann. 99 (1928), 134-141.
12. L. S. Pontryagin, *Nepreryvnye gruppy,* 2nd Edition, revised, Gostehizdat, Moskva, 1954.
13. Yu. A. Šreĭder, *Stroenie maksimal'nyh idealov v kol'cah co svertkoĭ,* Mat. Sbornik 27(69) (1950), 297-318.
14. N. Wiener, and R. H. Pitt, *On absolutely convergent Fourier-Stieltjes transforms,* Duke Math. J. 4 (1938), 420-436.

The University of Washington