A NOTE ON REGULAR GROUP RINGS

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The purpose of this note is to prove the following two theorems.

THEOREM 1. *If* R *is the group ring of a group* G *with respect to a ring* A *with identity, and if* R *is regular, then* G *is a torsion group, and* A *is uniquely divisible by the order of each element in* G.

THEOREM 2. *If* G *is a locally finite group and* A *is a regular ring with identity which is uniquely divisible by the order of every element in* G, *then the group ring of* G *with respect to* A *is regular.*

These theorems answer a question raised by I. Kaplansky. They have been proved by M. Auslander by the methods of homological algebra [1]. The proofs given here are quite elementary. Very similar proofs have been found independently by I. N. Herstein (private communication).

To prove Theorem 1, we may suppose that there is an $x \neq 1$ in G, and we set $u = 1 - x$ in R. Since R is regular, an s exists in R with $usu = u$. We put $t = 1 - su$ and observe that, since u has coefficient sum zero, $t \neq 1$. But $ut = u - usu = 0$, and therefore u is a left zero divisor. We may then write

$$(1 - x)\left(a_0 + \sum_1^N a_k g_k\right) = 0,$$

where the $a_i \in A$, $a_0 \neq 0$ and the g_k are distinct members of G different from 1. This equation may be rewritten as

$$a_0 + \sum_1^N a_k g_k - a_0 x - \sum_1^N a_k (x g_k) = 0,$$

and since no $xg_k = x$, it must happen that some g_k is x and the corresponding a_k is a_0. Renaming the g_k, if necessary, we may rewrite the equation as

$$a_0 + \sum_2^N a_k g_k - a_0 x^2 - \sum_2^N a_k (x g_k) = 0.$$

Now if $x^2 \neq 1$, we apply the same argument to obtain

$$a_0 + \sum_3^N a_k g_k - a_0 x^3 - \sum_3^N a_k (x g_k) = 0.$$

Received October 3, 1957.

This work was done while the author held a Summer Faculty Research Fellowship from the Rackham School of the University of Michigan.

We continue in this fashion, and either we arrive at an integer $M < N$ with $x^M = 1$, or else $g_k = x^k$ and $a_k = a_0$ $(k = 1, 2, \cdots, N)$. Since the second alternative implies $x^{N+1} = 1$, we have shown that x has finite order—G is a torsion group. We now go back to our original relation—$(1 - x)(a_0 + \Sigma_1^N a_k g_k) = 0$. Recall that the second factor is th, where $h \in G$ and t is as defined above. We now suppose x has order $M > 1$, and proceed as above. This time the process will yield (with suitable relabeling) $g_k = x^k$, $a_k = a_0$, for $k = 1, 2, \cdots, M - 1$; that is, $th = a_0(1 + x + \cdots + x^{M-1}) + t_1$, where t_1 is "shorter" than t, and t_1 is a right annihilator of u (since this is the case for $1 + x + \cdots + x^{M-1}$ and t). Then the same argument shows that for some $h_1 \in G$,

$$t_1 h_1 = b_0(1 + x + \cdots + x^{M-1}) + t_2,$$

where t_2 is "shorter" than t_1 and is a right annihilator of u. Continuing this process, we eventually arrive at the conclusion that $t = 1 + x + \cdots + x^{M-1})y$, for some $y \in R$. From the definition of t, it now follows that

$$1 = s(1 - x) + (1 + x + \cdots + x^{M-1})y .$$

Now we apply the coefficient sum homomorphism from R onto A, and observe that $M \cdot 1$ is a unit in A (A is uniquely divisible by the order of each element in G). This completes the proof of Theorem 1.

We shall merely outline the proof of Theorem 2, since it follows a well-established pattern. Clearly it is sufficient to consider the case of a finite group. Then the group ring is a finitely generated free module over the ring A, and every finitely generated left ideal is a finitely generated submodule. By [2], the left ideal has an A-module complement, and the unique divisibility hypothesis allows us to carry out the usual averaging process to obtain a left ideal complement. This in turn implies that the group ring is regular.

REFERENCES

1. M. Auslander, *On regular group rings*, Proc. Amer. Math. Soc. 8 (1957), 658-664.

2. J. v. Neumann, *Continuous geometry, Part II,* Princeton, 1937.

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