ITERATED LIMITS

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Let Q_1 and Q_2 be directed sets with order relations R_1 and R_2, respectively, and f a function on Q_1Q_2 to the reals. In this situation the following iterated limits theorem is well known (see Moore and Smith [6, p. 116]): *if* $\lim_{q_1} f(q_1q_2)$ *exists for every* q_2 *and* $\lim_{q_2} f(q_1q_2)$ *exists uniformly for* q_1 *on* Q_1, *then the iterated limits* $\lim_{q_1} \lim_{q_2} f(q_1q_2)$, $\lim_{q_2} \lim_{q_1} f(q_1q_2)$ *and the double limit* $\lim_{q_1q_2} f(q_1q_2)$ *all exist and are equal.* For the case where Q_1 and Q_2 are the positive integers in their natural order, that is, f_{mn} is a double sequence, and f_{mn} is monotone nondecreasing in n for each m, the uniformity condition is also necessary; in other words, if $\lim_m \lim_n f_{mn} = \lim_n \lim_m f_{mn}$, then the inner limits are both uniform, and the double limit exists (see Hildebrandt [4, p. 81]). This note gives the following generalization of this result to Moore-Smith directed limits.

THEOREM. *If* $f(q_1q_2)$ *is a real-valued function on* Q_1Q_2 *such that* $f(q_1q_2)$ *is monotone in* q_1 *in the sense that* $q_1'R_1q_1''$ *implies* $f(q_1'q_2) \geq f(q_1''q_2)$ *for every* q_2, *and if* $\lim_{q_1} \lim_{q_2} f(q_1q_2) = \lim_{q_2} \lim_{q_1} f(q_1q_2)$, *all limits being assumed to exist as finite numbers, then the double limit* $\lim_{q_1q_2} f(q_1q_2)$ *exists and is equal to the iterated limits.*

Let $\lim_{q_1} f(q_1q_2) = g(q_2)$ and $\lim_{q_2} f(q_1q_2) = h(q_1)$, and $\lim_{q_2} g(q_2) = \lim_{q_1} h(q_1) = a$. Then, because of the monotoneity of f in q_1, there exists for every $e > 0$ a q_{2e} such that $q_2R_2q_{2e}$ implies the relation

$$f(q_1q_2) \leq g(q_2) \leq a + 2e.$$

On the other hand, select q'_{1e} so that $h(q'_{1e}) \geq a - e$, and q'_{2e} so that $q_2R_2q_{2e}$ implies $f(q'_{1e}q_2) \geq h(q'_{1e}) - e$. Then, if $q_1R_1q'_{1e}$ and $q_2R_2q'_{2e}$, it follows from the monotoneity of f that

$$f(q_1q_2) \geq f(q'_{1e}q_2) \geq h(q'_{1e}) - e \geq a - 2e.$$

Consequently, if q''_{2e} is chosen so that $q''_{2e}R_2q_{2e}$ and $q''_{2e}R_2q'_{2e}$, we have that $q_1R_1q_{1e}$ and $q_2R_2q''_{2e}$ implies $a - 2e \leq f(q_1q_2) \leq a + 2e$; in other words, the double limit exists and has the desired value.

Since $\lim_{q_2} g(q_2) = a$, it follows further that for every $e > 0$ there exist q_{1e} and q_{2e} such that if $q_1R_1q_{1e}$ and $q_2R_2q_{2e}$, we have $|f(q_1q_2) - g(q_2)| \leq 2e$, a sort of pseudo-uniformity. In case Q_2 is the set of integers in their natural order, there are only finite number of $n \leq n_e$, for which of course $f(q_1n)$ converges to $g(n)$, so that we actually have uniformity as to n. Since Q_1 and Q_2 are interchangeable, here, we have

COROLLARY 1. *Under the hypothesis of the Theorem, if either* Q_1 *or* Q_2 *is the class of positive integers in their natural order, then the convergence of* $f(nq)$ *is uniform as to* n.

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This includes the theorem on double sequences mentioned at the beginning.

It might be noted that of necessity $h(q_1)$ is also monotone nondecreasing in q_1. Further observe that the proof does not require that $g(q_2) = \lim_{q_1} f(q_1q_2)$. It is sufficient that $\lim_{q_1} f(q_1q_2) \leq g(q_2)$ for all q_1. Hence we have

COROLLARY 2. *If* $f(q_1q_2)$ *is monotone nondecreasing in* q_1 *for each* q_2, *and if there exists a* $g(q_2)$ *such that* $\lim_{q_1} f(q_1q_2) \leq g(q_2)$ *for each* q_2 *with*

$$\lim_{q_2} g(q_2) = \lim_{q_1} \lim_{q_2} f(q_1q_2),$$

then the double limit $\lim_{q_1q_2} f(q_1q_2)$ *exists and is equal to these limits.*

The theorem leads to a simple proof of the well-known theorem of Dini: *If* $\{f_n(x)\}$ *is a sequence of continuous functions on* $[a, b]$, *monotone nondecreasing in* n *for each* x, *and if* $f(x) = \lim_n f_n(x)$ *is continuous, then the convergence is uniform on* $[a, b]$. Let x_0 be any point of $[a, b]$, and take Q_1 to be the integers in their natural order, while Q_2 consists of the real numbers x ordered so that $q_2'R_2q_2''$ means $0 < |x' - x_0| < |x'' - x_0|$. Then it is obvious that

$$\lim_n \lim_{x \to x_0} f_n(x) = \lim_{x \to x_0} \lim_n f_n(x).$$

Consequently, for every e there exists an n_{ex_0} and a d_{ex_0} such that if $n \geq n_{ex_0}$ and $0 < |x - x_0| \leq d_{ex_0}$, then $|f_n(x) - f(x_0)| \leq e$. Obviously, the condition $0 < |x - x_0|$ can be changed to $0 \leq |x - x_0|$. Also, because of the continuity of $f(x)$, we have $|f(x) - f(x_0)| \leq e$ whenever $|x - x_0| \leq d'_{ex_0}$, where d'_{ex_0} is suitably chosen. Consequently, for $n \geq n_{ex_0}$ and $|x - x_0| \leq d''_{ex_0}$, we have $|f_n(x) - f(x)| \leq 2e$. The Borel theorem now gives us a finite number of intervals $(x_i - d''_{ex_i}, x_i + d''_{ex_i})$ covering $[a, b]$, and consequently a finite number of n_{ex_i}, so that if n_e is the largest of these, then for $n \geq n_e$ and every x on $[a, b]$, we have $|f_n(x) - f(x)| \leq 2e$; this is the desired uniformity.

A similar procedure gives us the parity theorem: *If* $\{f_n(x)\}$ *is a sequence of monotone nondecreasing functions on* $[a, b]$ *converging to the continuous function* $f(x)$ *(obviously monotone nondecreasing), then the convergence is uniform on* $[a, b]$ (see Buchanan and Hildebrandt [3]).

Note that the functions $f_n(x)$ need not be continuous. Again, let x_0 be any point of $[a, b]$, and select Q_1 to be the $x < x_0$ in their natural order, and Q_2 the integers n in their natural order. Then, applying Corollary 2, we conclude that for every $e > 0$, there exists an n_{ex_0} and a d_{ex_0} such that, if $n \geq n_{ex_0}$ and

$$0 < x - x_0 \leq x' - x_0 = d_{ex_0},$$

we have $|f_n(x) - f(x_0)| \leq e$. Since Corollary 2 is also true if the order on x is reversed, and "monotone nondecreasing" is changed to "monotone nonincreasing," we conclude that for $n \geq n_{ex_0}$ and $0 < |x - x_0| \leq d_{ex_0}$ we have $|f_n(x) - f(x_0)| \leq e$. The conclusion then follows as in the theorem of Dini above.

The parity theorem can be extended if we note that if $\{f_n(x)\}$ is a sequence of functions of bounded variation, converging to $f(x)$ in such a manner that $\lim_n V_a^b f_n = V_a^b f$ (where $V_a^b f$ is the total variation of f on $[a, b]$), then $\lim_n V_a^x f_n = V_a^x f$ (see Adams and Clarkson [1, p. 414]). The parity theorem can then be applied to the positive and negative variations of the functions $f_n(x)$ and $f(x)$, so that we have: *If* $\{f_n(x)\}$ *is a sequence of functions of bounded variation such that* $f_n(x)$ *converges to* $f(x)$ *for every* x *and* $\lim_n V_a^b f_n = V_a^b f$, *and if* $f(x)$ *is continuous, then the convergence of* f_n *to* f *is uniform on* $[a, b]$.

These theorems give cases where uniform convergence is both necessary and sufficient that a sequence of continuous functions converge to a continuous function.

Another application of our fundamental theorem gives a proof of the following theorem on the convergence of a sequence of Stieltjes integrals; this theorem is a generalization, due to Shohat [8, p. 478], of a theorem of Bray [2, p. 180]: *if* $\{g_n(x)\}$ *is a bounded sequence of monotonic functions converging to* $g(x)$ *on a denumerable dense subset of* $[a, b]$ *including* a *and* b, *and if the Riemann-Stieltjes integrals* $\int f\,dg_n$ *and* $\int f\,dg$ *exist, then* $\lim_n \int f\,dg_n = \int f\,dg$ (see also Schwartz [7]).

To show this, we first prove a theorem relating to Riemann-Stieltjes σ-integrals, where the convergence of the approximating sums is according to set inclusion of the points of subdivision σ. We note that if $g(x)$ is monotone nondecreasing, a necessary and sufficient condition that $\sigma\int f\,dg$ exist is that $\lim_\sigma \Sigma_\sigma (\omega f \Delta g) = 0$ (see Hildebrandt [5, p. 271]). If we assume that $g_n(x)$ converges to $g(x)$, then $\lim_n \Delta g_n = \Delta g$. Hence, if $\int f\,dg_n$ and $\int f\,dg$ exist, then $\lim_\sigma \lim_n \Sigma_\sigma (\omega f \Delta g_n) = \lim_n \lim_\sigma \Sigma_\sigma (\omega f \Delta g_n) = 0$. Moreover, for each n, $\{\Sigma_\sigma \omega f \Delta g_n\}$ is monotone nonincreasing as to σ. Hence, by our theorem, there exists an n_e and a σ_e such that if $n \geq n_e$ and $\sigma \geq \sigma_e$, we have $\Sigma_\sigma \omega f \Delta g_n \leq e$. By Corollary 1, we conclude that $\Sigma_\sigma \omega f \Delta g_n$ converges to zero uniformly as to n. Since $|\int f\,dg - \Sigma_\sigma f \Delta g| \leq \Sigma_\sigma \omega f \Delta g$, it follows that the approximating sums $\Sigma_\sigma f \Delta g_n$ converge uniformly to $\int f\,dg_n$, so that by the iterated limits theorem we have $\lim_n \int f\,dg_n = \int f\,dg$. We thus have proved: *If* $\{g_n(x)\}$ *is a bounded sequence of monotone functions converging to* $g(x)$ *for every* x, *and if* $\sigma\int f\,dg_n$ *and* $\sigma\int f\,dg$ *exist, then* $\lim_n \sigma\int f\,dg_n = \sigma\int f\,dg$.

To obtain the Bray-Shohat theorem, we note that if for two functions f and g the Riemann-Stieltjes integral $RS\int f\,dg$ exists, where convergence is as to maximum length of the intervals of the subdivisions σ of $[a, b]$, then $\lim_\sigma \Sigma_\sigma \omega f \Delta g = 0$, where the points of σ need to be taken only from a denumerable dense set on $[a, b]$ including a and b. For the σ' so restricted, the uniform convergence of $\Sigma_{\sigma'} \omega f \Delta g_n$ to zero then follows, which in turn implies the theorem, since $\lim_{\sigma'} \Sigma_{\sigma'} f \Delta g = RS\int f\,dg$.

Obviously these theorems give rise to theorems where the $g_n(x)$ are functions of bounded variation, if conditions are added so that the $V_a^x g_n$ converge to $V_a^x g$ for suitable values of x.

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