AN ELEMENTARY PROOF OF A FUNDAMENTAL THEOREM IN THE THEORY OF BANACH ALGEBRAS

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INTRODUCTION

It is well known that the theory of Banach algebras rests on the Mazur-Gelfand theorem, which states that every complex normed division algebra is isomorphic to the complex field itself. This result, which is directly equivalent to the existence of a spectrum for elements of a normed algebra, was announced by Mazur [5] and proved by Gelfand [2], who used a generalization of the Liouville theorem to vector-valued functions. More recently, elementary proofs of the theorem have been given which avoid complex function theory by an ingenious use of roots of unity [4,7,8]. Another result, fundamental to the theory of Banach algebras and also due to Gelfand [2] in the general case, is the so-called "spectral radius formula," which states that if $\sigma(x)$ is the spectrum of an element x in a Banach algebra with norm $||x||$, then

$$\max_{\lambda \in \sigma(x)} |\lambda| = \lim_{n \to \infty} ||x^n||^{1/n}.$$

The quantity on the left is the spectral radius. Since the formula asserts the existence of a complex number in the spectrum of x with absolute value equal to $\lim ||x^n||^{1/n}$, it can be regarded as a precise statement concerning the existence of a spectrum. The usual proofs of the spectral radius formula also depend heavily on complex function theory. We give below an elementary proof which avoids function theory, and we obtain, incidentally, another elementary proof of the Mazur-Gelfand theorem. It involves roots of unity in a way similar to the proofs mentioned above; but the proof is different and is indeed considerably simpler, in spite of the greater precision of the result.

1. PRELIMINARIES

We assume that $\mathfrak{A}$ is a complex normed algebra. In other words, $\mathfrak{A}$ is an algebra over the complex field, and the vector space of $\mathfrak{A}$ is a normed linear space whose norm $||x||$ satisfies the multiplicative condition $||xy|| \leq ||x|| \, ||y||$. If $\mathfrak{A}$ is complete in the norm topology, then it is a Banach algebra. In order to deal with the case in which there is no identity element, it is convenient to use the "circle operation" $x \circ y = x + y - xy$, which is associative and has zero as an identity element. An element x is called *quasi-regular (quasi singular)* provided it has (does not have) an inverse, relative to the circle operation. This inverse, if it exists, is called the *quasi-inverse* of x and is denoted by x°. The set Q of all quasi-regular elements in $\mathfrak{A}$ is a group under the circle operation. An important and elementary property of normed algebras is that the mapping $x \to x^{\circ}$ of Q onto itself is continuous. Arens [1] gives a proof of the corresponding result for the regular elements in a normed algebra with an identity. His proof is adapted to the present case and included here

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for the sake of completeness. Let x and x + h be elements of Q, and set $(x+h)^o = x^o + k$. Then the problem is to show that if $||h||$ is small, then $||k||$ is also small. Since $(x+h)\circ(x^o+k) = 0$ and $x \circ x^o = 0$, we obtain

$$h(1 - x^o) = hk + xk - k,$$

where $h(1 - x^o)$ is written in place of $h - hx^o$ for convenience, and in spite of the fact that $\mathfrak{A}$ may not possess an identity element. An application of this relation, along with $x^o \circ x = 0$, gives $(1 - x^o)h(1 - x^o) = hk + x^o hk - k$. It follows directly from the norm inequalities that

$$||k|| - (1 + ||x^o||)\,||h||\,||k|| \leq (1 + ||x^o||)^2\,||h||.$$

Therefore, if $||h|| < \frac{1}{2}(1 + ||x^o||)^{-1}$, then $||k|| < 2(1 + ||x^o||)^2\,||h||$, and the desired continuity follows.

In an algebra with an identity, the spectrum of an element x is defined to consist of all complex numbers λ such that $x - \lambda$ is singular, that is, does not possess an inverse relative to multiplication. The following definition, due to Hille [3; p. 458], applies whether or not there is an identity element. When both definitions apply, the two spectra are equal, except possibly for the point $\lambda = 0$ which may belong to the first without belonging to the second spectrum.

The spectrum $\sigma(x)$ *of an element* x *in the complex algebra* $\mathfrak{A}$ *consists of all nonzero complex numbers* λ *such that* $\lambda^{-1}x$ *is quasi-singular, plus the point* $\lambda = 0$ *provided there does not exist in* $\mathfrak{A}$ *a nonzero idempotent* e *and an element* y *such that* $ex = xe = x$ *and* $xy = yx = e$.

Finally, Pólya and Szegö [6; p. 171, solution of Prob. 98] show by an elementary argument that, if $\{a_n\}$ is a sequence of real numbers such that $a_{m+n} \leq a_m + a_n$ for all m and n, then $\lim_{n\to\infty} a_n/n$ exists either as a finite number or as $-\infty$, and, moreover, the limit is equal to the greatest lower bound of the numbers

$$a_n/n \qquad (n = 1, 2, 3, \cdots).$$

An application of this result with $a_n = \log ||x^n||$ shows that $\lim_{n\to\infty} ||x^n||^{1/n}$ always exists and is equal to the greatest lower bound of the numbers

$$||x^n||^{1/n} \qquad (n = 1, 2, 3, \cdots).$$

2. THE SPECTRAL-RADIUS FORMULA

In the following discussion, the element x is fixed and the value of $\lim ||x^n||^{1/n}$ is denoted by ν. It is simple to verify that if $\mathfrak{A}$ is a Banach algebra, then the series $-\Sigma_{n=1}^{\infty} \lambda^{-n} x^n$ converges absolutely to an element of $\mathfrak{A}$, for each scalar λ with $|\lambda| > \nu$. Moreover, a straightforward calculation shows that the sum of this series is actually a quasi-inverse for $\lambda^{-1}x$. Thus, for Banach algebras, either $\sigma(x)$ is vacuous or the spectral radius of x is less than or equal to ν. The nonvacuity of $\sigma(x)$, together with the reverse inequality needed to establish the spectral radius formula, is given by the following theorem, which is valid for an arbitrary normed algebra.

THEOREM. *Let* x *be an element of any complex normed algebra* $\mathfrak{A}$. *Then there exists a complex number* λ *in the spectrum of* x *such that* $\lim ||x^n||^{1/n} \leq |\lambda|$.

Proof. If $0 \notin \sigma(x)$, then by definition there exists a nonzero idempotent e and an element y in $\mathfrak{A}$ such that $ex = xe = x$ and $xy = yx = e$. It follows that $e = x^n y^n$, and hence that $||e||^{1/n} \leq ||x^n||^{1/n} ||y||$ for all n. This implies $1 \leq \nu ||y||$. Therefore if $\nu = 0$, then $0 \in \sigma(x)$, and the desired result is true in this case. Next, assume that $\nu > 0$, and suppose that the theorem is false. Then the function $\phi(\lambda) = (\lambda^{-1}x)^o$ is defined and continuous for $|\lambda| \geq \nu$. Moreover, since $\lambda^{-1}x \to 0$ as $\lambda \to \infty$, continuity of the quasi-inverse implies that $(\lambda^{-1}x)^o \to 0$ as $\lambda \to \infty$. Therefore ϕ is uniformly continuous for $|\lambda| \geq \nu$. In the remainder of the discussion, λ will always be a complex number such that $|\lambda| \geq \nu$.

Let $\omega_1, \cdots \omega_n$ be the nth roots of unity, and for any complex number α write $\alpha_j = \alpha\omega_j$ $(j = 1, \cdots, n)$. Then the polynomial $1 - \lambda^{-n}\xi^n$ can be factored in the form $1 - \lambda^{-n}\xi^n = (1 - \lambda_1^{-1}\xi)(1 - \lambda_2^{-1}\xi) \cdots (1 - \lambda_n^{-1}\xi)$. Writing this relation in terms of the circle operation and substituting x for ξ, we obtain

$$\lambda^{-n} x^n = (\lambda_1^{-1} x) \circ (\lambda_2^{-1} x) \circ \cdots \circ (\lambda_n^{-1} x).$$

It follows that $\lambda^{-n}x^n$ is quasi-regular for each n. Next, set

$$R_j = -(\lambda_j^{-1} x + \lambda_j^{-2} x^2 + \cdots + \lambda_j^{-n+1} x^{n-1}),$$

and observe that $\lambda^{-n}x^n = (\lambda_j^{-1}x) \circ R_j$. Since $\lambda^{-n}x^n$ and $\lambda_j^{-1}x$ are quasi-regular, this can be rewritten in the form

$$\phi(\lambda_j) = R_j \circ (\lambda^{-n}x^n)^o. \tag{1}$$

Now each term of R_j is of the form $\omega_j^{-k} \lambda^{-k} x^k$, where $1 \leq k \leq n - 1$. Therefore $\Sigma R_j = 0$, and summation of (1) for $j = 1, 2, \cdots, n$ gives

$$\frac{1}{n} \sum_{j=1}^{n} \phi(\lambda_j) = (\lambda^{-n}x^n)^o. \tag{2}$$

Since ϕ is uniformly continuous, there exists for arbitrary $\varepsilon > 0$ a μ (independent of n) such that $\nu < \mu$ and $||\phi(\nu_j) - \phi(\mu_j)|| < \varepsilon$ for $j = 1, 2, \cdots, n$. Using equation (2), we obtain at once

$$||(\nu^{-n} x^n)^o - (\mu^{-n}x^n)^o|| < \varepsilon, \tag{3}$$

for all n. On the other hand, since $\nu < \mu$ it follows that $\mu^{-n} x^n \to 0$, and therefore $(\mu^{-n}x^n)^o \to 0$ as $n \to \infty$. Therefore (3) implies that $||(\nu^{-n}x^n)^o|| < \varepsilon$ for sufficiently large n. In other words, $(\nu^{-n}x^n)^o \to 0$ and hence $\nu^{-n}x^n \to 0$ as $n \to \infty$. However, this is impossible, since $||\nu^{-n} x^n|| \geq 1$ for all n. Therefore the supposition that there is no λ in $\sigma(x)$ with $|\lambda| \geq \nu$ is false, and the proof is complete.

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