ON POWER SERIES, AREA, AND LENGTH

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Let C be the unit circle, and U the open unit disk in the complex plane. Denote by Φ the class of functions that are holomorphic in U and map every disk that is internally tangent to C onto a Riemann configuration of infinite area, and by Λ the class of functions that are holomorphic in U and map every rectilinear segment in U that terminates in a point of C onto a curve of infinite length. Lohwater and Piranian have established [1] the existence of functions in Φ of the form $\Sigma a_k z^k$ with $\Sigma |a_k| < \infty$. The first one of the two theorems which we shall prove implies that, in a certain sense, most functions of this form belong to Φ; we are indebted to Karl Zeller for suggesting a demonstration that is more elementary than our original one. No function $\Sigma a_k z^k$ with $\Sigma |a_k| < \infty$ belongs to Λ, however, because such a function maps every radius of U onto a curve whose length is not greater than $\Sigma |a_k|$. Our second theorem asserts that, for every $p > 1$, "most" functions of the form $\Sigma a_k z^k$ with $\Sigma |a_k|^p < \infty$ belong to Λ.

For $p \geq 1$, denote by $\mathfrak{L}_p$ the Banach space of all complex sequences $\{a_k\}$ for which $\Sigma |a_k|^p < \infty$; $\|\{a_k\}\| = (\Sigma |a_k|^p)^{1/p}$. With the element $\{a_k\}$ of $\mathfrak{L}_p$, associate the function $\Sigma a_k z^k$. If, as $j \to \infty$, the elements $\{a_k^{(j)}\} \in \mathfrak{L}_p$ converge to $\{a_k\} \in \mathfrak{L}_p$, then the sequence of functions $\Sigma a_k^{(j)} z^k$ converges uniformly to $\Sigma a_k z^k$ on every compact subset of U; this well-known fact is used implicitly in proving below that certain sets, E_m and $F_m(n)$, are closed.

A convex region D is called a tangential domain, if it lies in U, the intersection of its closure and C is the point 1, and the only straight line through the point 1 that does not intersect D is the tangent to C. Let Φ_D be the class of functions that are holomorphic in U and, for every real θ, map the region $D_\theta = \{ze^{i\theta} : z \in D\}$ onto a Riemann configuration of infinite area. Piranian and Rudin have proved [2, Theorem 4] that for every tangential domain D there exists a function in Φ_D of the form $\Sigma a_k z^k$ with $\{a_k\} \in \mathfrak{L}_1$; let R_D be the set of all elements of $\mathfrak{L}_1$ whose associated functions belong to Φ_D. If the boundary of D has infinite curvature at the point 1, then $\Phi_D \subset \Phi$.

THEOREM 1. *For every tangential domain* D, R_D *is a residual subset of* $\mathfrak{L}_1$.

Proof. For every natural number m, define E_m to be the set of those elements of $\mathfrak{L}_1$ whose associated functions do not map D_θ for every θ onto a Riemann configuration of area greater than m. Since C is compact, E_m is closed.

Suppose that P(z) is a polynomial and $t > 0$. For any θ, let G_n $(n = 2, 3, 4, \cdots)$ be the intersection of D_θ with the annulus $1 - 1/n < |z| < 1 - 1/2n$. Since D_θ is a tangential domain, the area of G_n is $g(n)/n^2$, where $g(n) \to \infty$ as $n \to \infty$; moreover, in G_n the modulus of the derivative of tz^n is greater than tn/e. Consequently, if n is sufficiently large, the function $P(z) + tz^n$ maps G_n, and hence D_θ for every θ, onto a Riemann configuration of area greater than m. Given $\varepsilon > 0$ and $\{a_k\} \in \mathfrak{L}_1$, choose K so large that $\Sigma_{k=K+1}^{\infty} |a_k| < \varepsilon/2$, set $P(z) = \Sigma_{k=0}^{K} a_k z^k$, let $t = \varepsilon/2$, and take n to be greater than K and so large that, if $b_k = a_k$ $(k = 0, 1, \cdots, K)$, $b_k = \varepsilon/2$ for $k = n$, and $b_k = 0$ for all other nonnegative integers k, we have $\{b_k\} \in \mathfrak{L}_1 - E_m$; clearly, $\|\{a_k\} - \{b_k\}\| < \varepsilon$.

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Thus, E_m is nowhere dense in $\mathfrak{L}_1$, and hence $R_D = \mathfrak{L}_1 - \bigcup E_m$ is a residual subset of $\mathfrak{L}_1$.

If $p > 1$, let R_p be the set of all elements of $\mathfrak{L}_p$ whose associated functions belong to Λ.

THEOREM 2. *For every* $p > 1$, R_p *is a residual subset of* $\mathfrak{L}_p$.

Proof. For every natural number n, let $F_m(n)$ $(m = 1, 2, 3, \cdots)$ be the set of those elements of $\mathfrak{L}_p$ whose associated functions do not map every rectilinear segment that lies in the annulus $1 - 1/n \leq |z| < 1$ and extends from $|z| = 1 - 1/n$ to C, onto a curve of length greater than m. Since C and the circle $|z| = 1 - 1/n$ are compact, $F_m(n)$ is closed.

Suppose that $\varepsilon > 0$, m is a natural number, and $\{a_k\} \in \mathfrak{L}_p$. Choose J so large that $\sum_{j=J+1}^{\infty} j^{-p} < (\varepsilon/2)^p$, r so large that $\sum_{t=1}^{r} (J + t)^{-1} > 4em$, and K so large that $\sum_{k=K+1}^{\infty} |a_k|^p < (\varepsilon/2)^p$ and $|a_k| < (J + r)^{-1}$ $(k = K + 1, K + 2, K + 3, \cdots)$. If

$$f(z) = \sum_{k=0}^{K} a_k z^k + \sum_{j=1}^{r} c_j z^{n_j} \qquad (\max(K, n) < n_1 < n_2 < \cdots < n_r),$$

where $|c_j| = (J + j)^{-1}$ $(j = 1, 2, \cdots, r)$, and if the natural numbers n_j $(j = 1, 2, \cdots, r)$ are sufficiently large and far apart, then

$$|f'(z)| > n_j |c_j|/2e \qquad (1 - 1/n_j \leq |z| \leq 1 - 1/2n_j;\ j = 1, 2, \cdots, r),$$

and consequently f(z) maps every rectilinear segment that lies in the annulus $1 - 1/n \leq |z| < 1$ and extends from $|z| = 1 - 1/n$ to C onto a curve of length greater than

$$\sum_{j=1}^{r} \frac{1}{2n_j} \cdot \frac{n_j |c_j|}{2e} = \frac{1}{4e} \sum_{j=1}^{r} |c_j| > m.$$

If, further, $\arg c_j = \arg a_{n_j}$ for every $j = 1, 2, \cdots, r$ for which $a_{n_j} \neq 0$, and we set $d_k = a_k$ $(k = 0, 1, \cdots, K)$, $d_k = c_j$ for $k = n_j$ $(j = 1, 2, \cdots, r)$, and $d_k = 0$ for all other nonnegative integers k, then $\{d_k\} \in \mathfrak{L}_p - F_m(n)$ and $\|\{a_k\} - \{d_k\}\| < \varepsilon$.

Thus, $F_m(n)$ is nowhere dense in $\mathfrak{L}_p$, and hence $R_p(n) = \mathfrak{L}_p - \bigcup_m F_m(n)$ is a residual subset of $\mathfrak{L}_p$, so that $R_p = \bigcap_n R_p(n)$ is also a residual subset of $\mathfrak{L}_p$.

Remark 1. Let Λ^* denote the class of functions that are holomorphic in U and map every continuous curve in U that approaches C, onto a curve of infinite length. A slight extension of part of the foregoing argument shows that, if $\Sigma |c_k| = \infty$ and $\{n_k\}$ is an increasing sequence of natural numbers for which $n_k \to \infty$ fast enough as $k \to \infty$, then $\Sigma c_k z^{n_k} \in \Lambda^*$. If, in particular, we take $c_k = 1/k$, we see that there exists a sequence $\{a_n\}$ such that $\Sigma a_n z^n \in \Lambda^*$ and $\{a_n\} \in \mathfrak{L}_p$ for every $p > 1$. (See [4, p. 194], [3, p. 237] for related results.)

Remark 2 (by G. Piranian). Let Λ^{**} denote the class of functions

$$\Sigma a_k z^k \quad (\{a_k\} \in \mathfrak{L}_1)$$

which map every arc of C onto a nonrectifiable curve. A proof somewhat similar to that of Theorem 2 shows that there exists a residual subset R of $\mathfrak{L}_1$ such that, for every $\{a_k\} \in$ R, the associated function $\Sigma a_k z^k \in \Lambda^{**}$.

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