A THEOREM ON TWO-DIMENSIONAL VECTOR SPACES

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In classical projective geometry, homogeneous coordinates for the line are customarily introduced by means of an algorithm. If one wished to give a formal definition, one might begin by observing that projective lines can be manufactured from two-dimensional vector spaces in a natural way; then a system of homogeneous coordinates for a line L in a projective space might possibly be defined as a one-to-one mapping, from L onto a line so constructed, which preserves projectivities.

More specifically, if V is a two-dimensional vector space over a division ring D, let Π_V be the family of all lines of V which pass through the origin; and for any nonzero vector v of V, let [v] be the unique member of Π_V to which v belongs. A map p: $\Pi_V \to \Pi_V$ will be called a *projectivity* if there is some nonsingular linear transformation $\alpha: V \to V$ such that $[v]p = [v\alpha]$ for all $v \in V$. Then, if L is a line in a projective space P, the map h constitutes a *system of homogeneous coordinates* for L provided, for some vector space V over a division ring D, the map h: $L \to \Pi_V$ is one-to-one onto and p: $\Pi_V \to \Pi_V$ is a projectivity if and only if hph^{-1} is a projectivity of L (where projectivities of L are defined, as classically, to be sequences of perspectivities in P).

The question arises whether such a system of homogeneous coordinates is necessarily equivalent to the one given by the classical algorithm. Put algebraically, this question becomes: if V and W are two-dimensional vector spaces over division rings D and E, respectively, and if f: $\Pi_V \to \Pi_W$ is a one-to-one onto map which preserves projectivities, does there exist a semilinear isomorphism from V onto W which induces f? The map f induces a special isomorphism from the projective group of V onto the projective group of W, and a classical result due to Schreier and van der Waerden [5] tells us that if D and E are commutative and contain more than five elements, then *any* isomorphism between these groups yields an isomorphism of D onto E. Once we know that D and E are isomorphic, then Hua's determination of the automorphisms of the two-dimensional projective groups [4] yields the fact that f is indeed induced by a semilinear isomorphism of V onto W.

We shall show, below, that in general the map f induces either an isomorphism or an anti-isomorphism of D onto E and then, again by Hua's result, f is induced either by a semilinear isomorphism of V onto W, or by a semilinear isomorphism of V onto W* (the dual space of W), followed by the canonical map from W* to W.

We emphasize that our isomorphism of the projective group of V onto the projective group of W is a special one; and whether or not an arbitrary isomorphism yields an isomorphism or anti-isomorphism of D onto E remains an open question.

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THEOREM. *Let* V *and* W *be two-dimensional vector spaces over division rings* D *and* E, *respectively, and suppose* f: $\Pi_V \to \Pi_W$ *is one-to-one onto. Suppose further that if* G *and* H *denote the respective projective groups, then the map* f*: $G \to H$

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given by $pf^* = f^{-1}pf$ *is an isomorphism. Then either* D *is isomorphic to* E, *or* D *is anti-isomorphic to* E.

Proof. Choose bases $\{v_1, v_2\}$, $\{w_1, w_2\}$ for V, W respectively so that

$$[v_1]f = [w_1], \quad [v_2]f = [w_2], \quad [v_1 + v_2]f = [w_1 + w_2].$$

This defines a map $\sigma: D \to E$ given by $[v_1 + xv_2]f = [w_1 + x^\sigma w_2]$. Clearly, σ is one-to-one onto and $0^\sigma = 0$, $1^\sigma = 1$.

With respect to these bases, linear transformations (and hence projectivities) are represented by matrices from D_2 and E_2. Recall that two matrices represent the same projectivity if and only if one is a nonzero central scalar multiple of the other. Our first observation is that if $p \in G$ is represented by a matrix all of whose entries come from the center of D, then any matrix representing pf^* has all of its entries in the center of E. For suppose p is represented by (z_{ij}), where the elements z_{ij} are in the center of D; then p commutes with all members of G which leave $[v_1]$, $[v_2]$, and $[v_1 + v_2]$ fixed. Hence pf^* commutes with all members of H which leave $[w_1]$, $[w_2]$, and $[w_1 + w_2]$ fixed, and this in turn implies that if (x_{ij}) represents pf^*, then for each $u \neq 0$ in E there exists a z in the center of E such that $ux_{ij}u^{-1} = zx_{ij}$. In other words, the subfield of E generated by its center and the element x_{ij} is setwise invariant under all inner automorphisms of E, and by the Cartan-Brauer-Hua Theorem ([1], [3]) it is contained in the center of E—in particular, x_{ij} is in the center of E.

Our aim is to show that the map σ defined above is either an isomorphism or an anti-isomorphism. We begin by showing that σ is additive. Suppose $p \in G$ is represented by $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$; then pf^* is represented by a matrix of the form $\begin{pmatrix} f(a) & g(a) \\ 0 & h(a) \end{pmatrix}$, and invoking the relation $pf^* = f^{-1}pf$ we obtain, for all $x \in D$,

$$(a + x)^\sigma = f^{-1}(a)\, g(a) + f^{-1}(a)\, x^\sigma h(a).$$

Setting $x = 0$, we have $g(a) = f(a)a^\sigma$; and setting $a = 1$, we have

$$(1 + x)^\sigma = 1 + x^\sigma f^{-1}(1)\, h(1).$$

Now p has $[v_2]$ as its only fixed point, and therefore pf^* has $[w_2]$ as its only fixed point. Hence $f^{-1}(1)\, h(1) = 1$, and we have $(1 + x)^\sigma = 1 + x^\sigma$ for all $x \in D$. This in turn yields $f(a) = h(a)$ for all $a \in D$. If we denote our projectivity by p_a, and t is any projectivity leaving $[v_1]$ and $[v_2]$ fixed, then $tp_at^{-1} = p_b$, for some $b \in D$. This must carry over to Π_W, and computation with the corresponding matrices yields the fact that f(a) lies in the center of E. Now we can argue as before about the fixed points of p_a, and obtain $(a + x)^\sigma = a^\sigma + x^\sigma$ for all a and x in D.

Now suppose p is represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This matrix has entries from the center of D, and p interchanges $[v_1]$ and $[v_2]$ while leaving $[v_1 + v_2]$ fixed; $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a matrix which represents pf^*. Since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix},$$

it is clear that if g is represented by $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, then gf* is represented by $\begin{pmatrix} 1 & 0 \\ a^\sigma & 1 \end{pmatrix}$. Finally,

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 - a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & a^2 - a \\ 0 & 1 \end{pmatrix},$$

and if p is represented by the matrix on the left, pf* is represented by

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 - a^\sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (a^{-1})^\sigma & 1 \end{pmatrix} \begin{pmatrix} 1 & (a^2)^\sigma - a^\sigma \\ 0 & 1 \end{pmatrix}.$$

Multiplication and the observation that pf* leaves $[w_2]$ fixed give the result that pf* is represented by $\begin{pmatrix} (a^\sigma)^{-1} & 0 \\ 0 & a^\sigma \end{pmatrix}$. Since $pf^* = f^{-1}pf$, we obtain $(axa)^\sigma = a^\sigma x^\sigma a^\sigma$, for all a and x in D. A theorem of Hua's [2] then tells us that σ is either an isomorphism or an anti-isomorphism. This completes the proof of the theorem.

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