ON THE ACTION OF THE CIRCLE GROUP

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1. INTRODUCTION

The object of this note is to complement a recent series of discussions ([4], [5], [6]) of the operation of the circle group as a group of transformations. We shall consider transformation groups of the form (S^1, X), where X is locally compact, locally connected and separable metric. Associated with each (S^1, X) are two spaces in whose rational cohomology groups we shall be interested: the set $F \subset X$ of stationary points of (S^1, X), and the orbit space X/S^1. Smith ([2], [9]) has established a number of theorems about the operation of the integers mod p, Zp, as a group of transformations. These theorems concern the cohomology structure of the associated spaces, and the coefficients Zp are used. We shall establish the analogous theorems about the rational cohomology structure under an action of the circle group. Actually, by making use of the cohomology ring structure in our arguments, we are able in some places to avoid finite dimensionality restrictions on X. We are principally interested in illustrating techniques which may prove of value in later studies of compact transformation groups.

2. PRELIMINARIES

We denote by $H^i(X; Q)$ the rational Alexander-Wallace-Spanier cohomology groups of X, and by $H^i_c(X; Q)$ those cohomology groups which are based on cochains with compact support ([3]). Most of our arguments employ spectral sequences, and we shall assume familiarity with relevant techniques; for more details, we refer to ([1], [3]).

We recall that there is a canonical transformation group of the form (S^1, S^{2i+1}) such that S^{2i+1}/S^1 is the complex projective space P(i). Indeed, there is a whole sequence of such operations and a natural equivariant imbedding making an ascending sequence,

$$p_i: (S^1, S^{2i+1}) \to (S^i, S^{2(i+1)+1});$$

for if we form $(S^1, \bigcup_i S^{2i+1})$, we obtain the universal bundle for S^1, where $\bigcup_i S^{2i+1}$ is given the CW-topology ([1, p. 165]). We say that a transformation group (S^1, X) is *almost free* provided every isotropy subgroup is finite. For every almost free transformation group we form $(S^1, Y(i))$, where $Y(i) = S^{2i+1} \times X$ and S^1 acts on this cartesian product by diagonal action. Consider the natural diagram

$$\begin{array}{ccc} & (S^{2i+1} \times X)/S^1 & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ P(i) & & X/S^1 \end{array}$$

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Since S^1 acts freely on S^{2i+1}, the map $\alpha: (S^{2i+1}\times X)/S^1 \to P(i)$ is a fibration with fibre X. There is a spectral sequence with initial term

$$E_2^{s,t} \approx H^s(P(i); H_c^t(X; Q)),$$

whose E_∞-term is associated with $H_c^*(Y(i); Q)$. The map $\beta: (S^{2i+1}\times X)/S^1 \to X/S^1$ is not a fibration. For a point $((x)) \in X/S^1$, the inverse image $\beta^{-1}((x))$ is topologically S^n/G_x, where G_x is the isotropy group at $x \in X$. By definition, G_x is finite, so that $H^j(S^{2i+1}/G_x; Q) = 0$ for $j \leq 2i$. Since β is a proper map, we apply the Vietoris-Begle mapping theorem ([3, 21-05]) to conclude that

$$\beta^*: H^j(X/S^1; Q) \approx H^j(Y(i)/S^1; Q) \qquad (j \leq 2i),$$

$$\beta^*: H_c^j(X/S^1; Q) \approx H_c^j(Y(i)/S^1; Q) \qquad (j \leq 2i).$$

An immediate corollary is

LEMMA 2.1. *If* (S^1, X) *is almost free, there is a spectral sequence, with*

$$E_2^{s,t} \simeq H^s(P(i); H_c^t(X; Q)),$$

such that $E_\infty^j \simeq H_c^j(X/S^1; Q)$ for $j \leq 2i$.

This is a variation on a spectral sequence of Borel ([1, p. 179]). Since $\pi: Y(i) \to Y(i)/S^1$ is a principal fibration by the circle group, we can construct a Gysin sequence of the form

$$\to H_c^j(Y(i)/S^1; Q) \xrightarrow{\sigma} H_c^{j+2}(Y(i)/S^1; Q) \xrightarrow{\pi} {}^*H_c^{j+2}(Y(i); Q) \xrightarrow{\tau} H_c^{j+1}(Y(i)/S^1; Q),$$

where the homomorphism

$$\sigma: H_c^j(Y(i)/S^1; Q) \to H_c^{j+2}(Y(i)/S^1; Q)$$

is of the form $\sigma(e^j) = h^2 \cup e^j$, with $h^2 \in H^2(Y(i)/S^1; Q)$. Since

$$\beta^*: H^2(X/S^1; Q) \simeq H^2(Y(i)/S^1; Q),$$

we obtain an element $c^2 = \beta^{*-1}(h^2)$ in $H^2(X/S^1; Q)$ which is independent of i.

LEMMA 2.2. *If* (S^1, X) *is almost free and* X *is locally connected, then there exists a Gysin sequence of the form*

$$H_c^j(X/S^1; Q) \xrightarrow{\sigma} H_c^{j+2}(X/S^1; Q) \xrightarrow{\pi^*} H_c^{j+2}(X; Q) \xrightarrow{\tau} H_c^{j+1}(X/S^1; Q) \to,$$

where $\sigma(e^j) = c^2 \cup e^j$.

In general, we are saying that, with respect to rational cohomology, an almost free transformation group (S^1, X) is the same as a principal fibration. The next lemma is a generalization of the proposition that if X is compact and $e^r \in H^r(X, Q)$ for some $r > 0$, then for some integer n, $(e^r)^n = 0$.

LEMMA 2.3. *Let* X *be a locally compact metric space; then for any two cohomology classes* $e^r \in H_c^r(X, Q)$ *and* $h^s \in H^s(X; Q)$ $(s > 0)$, *there is an integer* n *such that* $e^r \cup (h^s)^n = 0$.

For a locally compact space, $H_c^r(X; Q) \simeq \text{dir}\varinjlim H^r(X, X - V; Q)$, where the direct limit is taken over all open sets $V \subset X$ with $\overline{V}$ compact. Let $f^r \in H^r(X, X - V; Q)$ be a representative of e^r. We have a natural cup product

$$H^r(X, X - V; Q) \otimes H^s(X; Q) \to H^{r+s}(X, X - V; Q).$$

(In the previous terminology, this product can be described by merely noting that $H^r(X, X - V; Q) \simeq H_c^r(V; Q)$). Let $C \supset V$ be a compact subset of X; then, by excision, $H^r(X, X - V; Q) \simeq H^r(C, C - V; Q)$. Let $i^*: H^s(X; Q) \to H^s(C; Q)$ be the homomorphism induced by the injection of C into X. By assumption, $f^r \cup h^s \in H^{r+s}(X, X - V; Q)$ is not zero. Now, from the natural diagram

$$\begin{array}{ccc} H^r(X, X - V; Q) \otimes H^s(X; Q) & \to & H^{r+s}(X, X - V; Q), \\ \wr\wr \qquad\qquad \downarrow i^* & & \wr\wr \\ H^r(C, C - V; Q) \otimes H^s(C; Q) & \to & H^{r+s}(C, C - V; Q) \end{array}$$

it follows that $f^r \cup i^*(h^s) \neq 0$. Since $i^*(h^s)$ is a cohomology class on a compact space, there exists an integer n such that $(i^*(h^s))^n = 0$; this implies that $f^r \cup (h^s)^n = 0$, or finally that $e^r \cup (h^s)^n = 0$, which is the assertion of the lemma.

3. THE ACTION OF S^1 ON A PEANO CONTINUUM

THEOREM 3.1. *If* (S^1, X) *is a transformation group on a Peano continuum, and if, for some integer* n, $H^t(X; Q) = 0$ *when* $t \geq n$, *then* $H^t(F; Q) = 0$ *when* $t \geq n$.

Recall that $F \subset X$ is the set of stationary points. Consider $(S^1, X - F)$, which if almost free. Since $H^t(X; Q) = 0$ for $t \geq n$, it follows that

$$\delta^*: H^t(F; Q) \simeq H^{t+1}(X, F; Q) \qquad (t \geq n),$$

and therefore the homomorphism

$$\pi^*: H_c^t((X - F)/S^1; Q) \to H_c^t(X - F; Q) \qquad (t \geq n + 1)$$

is onto. Therefore, in the Gysin sequence of Lemma 2.2,

$$\sigma: H_c^t((X - F)/S^1; Q) \to H_c^{t+2}((X - F)/S^1; Q)$$

is a monomorphism for $t \geq n$. Thus, if $H_c^n((X - F)/S^1; Q) \neq 0$, then $(c^2)^j \cup e^n \neq 0$ for all $e^n \in H_c^n((X - F)/S^1; Q)$, and this contradicts Lemma 2.3. Since $H_c^t((X - F)/S^1; Q) = 0$ for $t \geq n$, $H_c^t(X - F; Q) = 0$ for all $t \geq n + 1$. From the exact sequence of the pair (X, F) it follows that $H^t(F; Q) = 0$ for $t \geq n$.

COROLLARY 3.1. *If* (S^1, X) *is a transformation group on a Peano continuum, and if, for some integer* n, $H^t(X; Q) = 0$ $(t \geq n)$, *then* $H^t(X/S^1; Q) = 0$ $(t \geq n - 1)$.

This follows immediately from the consideration of the exact sequence of $(X/S^1, F)$.

THEOREM 3.2. *If* (S^1, X) *is a transformation group on a Peano continuum, and if, for some integer* n, $H^t(X; Q) = 0$ $(t \geq n)$, *then*

$$\sum_{t=0}^{\infty} \dim H^{2t}(F; Q) \leq \sum_{t=0}^{\infty} \dim H^{2t}(X; Q),$$

$$\sum_{t=0}^{\infty} \dim H^{2t+1}(F; Q) \leq \sum_{t=0}^{\infty} \dim H^{2t+1}(X; Q).$$

These inequalities are analogous to those found in ([7]), where Zp acts simplicially. They are obtained from the exact sequence of the pair

$$((S^{2i+1} \times X)/S^1, (S^{2i+1} \times F)/S^1),$$

where $2i + 1 >> n$. In this exact sequence,

$$H^t(X/S^1, F; Q) \simeq H^t((S^{2i+1} \times X)/S^1, P(i) \times F; Q)$$

for $t \leq 2i$. Thus, by 3.1,

$$H^t((S^{2i+1} \times X)/S^1; Q) \simeq H^t((P(i) \times F; Q),$$

where $n + 1 \leq t \leq 2i$. As observed previously, there is a spectral sequence associated with α: $(S^{2i+1} \times X)/S^1 \to P(i)$, with initial term

$$E_2^{s,t} \simeq H^s(P(i); H^t(X; Q)),$$

whose E_∞- term is associated with $H^*((S^{2i+1} \times X)/S^1; Q)$. For $n + 1 \leq t \leq 2j$,

$$\dim E_2^t = \sum_{t-p=2q} \dim H^p(X; Q) \geq \dim E_\infty^t = \sum_{t-p=2q} \dim H^p(F; Q).$$

Since $H^t(X; Q) = H^t(F; Q) = 0$ for $t \geq n$ and $2i >> n$, we can obtain the inequalities by taking t first even, then odd.

COROLLARY 3.2. *If* (S^1, X) *is a transformation group on a Peano continuum* X *such that* $\dim H^t(X; Q) < \infty$ $(t \geq 0)$, *and if, for some* n, $H^t(X; Q) = 0$ $(t \geq n)$, *then*

$$\dim H^t(F; Q) < \infty \qquad (t \geq 0),$$

$$\chi(F; Q) = \chi(X; Q).$$

Next we wish to show that cohomology local connectedness over the rationals is a property inherited by the set of stationary points. Our argument will illustrate the application of spectral sequence techniques to the determination of the cohomology local connectedness of a space. By the statement that the space X is n-clc at $x \in X$, we shall mean that for every closed neighborhood U_x of x there is a closed neighborhood $V_x \subset U_x$ such that the homomorphism

$$i^*: \tilde{H}^n(U_x; Q) \to \tilde{H}^n(V_x; Q)$$

is trivial, where i^* is induced by the inclusion, and where reduced groups are understood. A space is clc^n at x if it is m-clc for all $m \leq n$. We shall say that a space is n-clc or clc^n if it has the corresponding property at each point.

THEOREM 3.3. *If* (S^1, X) *is a transformation group on a finite-dimensional Peano continuum, and if* X *is* clc^∞ *over* Q, *then* $F \subset X$ *is also* clc^∞ *over* Q.

By finite-dimensional we refer only to the rational Alexandroff cohomology dimension. Our technique requires this assumption of finite dimensionality, and it seems likely to us that the assumption is unavoidable. Let $\dim_Q X = n$.

Recall the spectral sequence in Lemma 2.1. It may be replaced ([3, p. 181]) by a spectral sequence with

$$E_2^{s,t} \simeq H^s(P(\infty); H_c^t(X; Q))$$

such that $E_\infty^n \simeq H_c^n(X/S^1; Q)$, where (S^1, X) is almost free, and where $P(\infty)$ denotes the infinite-dimensional complex projective space. Therefore we may, for any transformation group (S^1, X) on a continuum, write the exact sequence

$$\to H^t(X/S^1, F; Q) \to H^t((E\times X)/S^1; Q) \to H^t(P(\infty)\times F; Q) \to,$$

where $E = \bigcup_i S^{2i+1}$ is the universal bundle for S^1. This convention simplifies our arguments.

We shall first show that it is sufficient to prove that for each $x \in F$ and for each closed neighborhood U_x in X, there exists a closed neighborhood V_x in X such that the homomorphism

$$i^*: H_c^t(U_x - U_x \cap F; Q) \to H_c^t(V_x - V_x \cap F; Q)$$

is trivial for all $t \geq 0$. Indeed, suppose that the existence of such a V_x has been established. There exists a closed neighborhood $W_x \subset V_x$ such that the homomorphism

$$I_1^*: H_c^t(V_x - V_x \cap F; Q) \to H_c^t(W_x - W_x \cap F; Q)$$

is also trivial. Now we make our choices so that

$$I_2^*: H^t(U_x; Q) \to H^t(V_x; Q),$$

$$I_3^*: H^t(V_x; Q) \to H^t(W_x; Q)$$

are both trivial, and we consider the natural diagram

$$\begin{array}{ccccc}
\tilde{H}^t(U_x; Q) & \to & \tilde{H}^t(U_x \cap F; Q) & \to & H_c^{t+1}(U_x - U_x \cap F; Q) . \\
\downarrow I_2^* & & \downarrow I_4^* & & \downarrow I^* \\
\tilde{H}^t(V_x; Q) & \xrightarrow{i^*} & \tilde{H}^t(V_x \cap F; Q) & \to & H_c^{t+1}(V_x - V_x \cap F; Q) \\
\downarrow I_3^* & & \downarrow I_5^* & & \downarrow I_1^* \\
\tilde{H}^t(W_x; Q) & \to & \tilde{H}^t(W_x \cap F; Q) & \to & H_c^{t+1}(U_x - U_x \cap F; Q)
\end{array}$$

Since $\operatorname{im} I_4^* = \operatorname{im} i^*$ and I_3^* is trivial, it follows that $I_5^* I_4^*$ is also trivial.

LEMMA 3.1. *Let* (S^1, X) *be a transformation group on a locally compact, locally connected space* X, *with* $\dim_Q X = n$, *and let* $U_1 \supset \cdots \supset U_{n+2}$ *be a descending chain of invariant continua in* X *such that the homomorphisms*

$$I_i^*: H^t(U_i; Q) \to H^t(U_{i+1}; Q)$$

are trivial for all $t \geq 0$; *then the homomorphism*

$$I^*: H_c^t(U_1 - U_1 \cap F; Q) \to H_c^t(U_{n+2} - U_{n+2} \cap F; Q)$$

is also trivial.

Since there will be no confusion, we shall use the notation

$$I_i^*: H_c^t(U_i - U_i \cap F; Q) \to H_c^t(U_{i+1} - U_{i+1} \cap F; Q).$$

Let $\hat{U}_i = U_i/S^1$. We identify $F \subset X$ with $F \subset X/S^1$, and note that if

$$\pi_{i+1}^*: H_c^t(\hat{U}_{i+1} - \hat{U}_{i+1} \cap F; Q) \to H_c^t(U_{i+1} - U_{i+1} \cap F; Q)$$

is the natural map, then $\operatorname{im} \pi_{i+1}^* \supset \operatorname{im} I_i^*$.

We have a collection of spectral sequences $\{{}_iE_r^{s,t}\}$ and homomorphisms $I_i^r: {}_iE_r^{s,t} \to {}_{i+1}E_r^{s,t}$ such that

$${}_iE_2^{s,t} \simeq H^s(P(\infty); H_c^t(U_i - U_i \cap F; Q)),$$

and such that ${}_iE_\infty$ is associated with $H_c^*(\hat{U}_i - \hat{U}_i \cap F; Q)$ and I_i^∞ is associated with

$$I_i^*: H_c^*(\hat{U}_i - \hat{U}_i \cap F; Q) \to H_c^*(\hat{U}_{i+1} - \hat{U}_{i+1} \cap F; Q).$$

The image of $I_i^2: {}_iE_2^{s,t} \to {}_{i+1}E_2^{s,t}$ consists entirely of permanent cocycles, since $\operatorname{im} \pi_{i+1}^* \supset \operatorname{im} I_i^*$ ([1, p. 179]). Let us suppose that

$$I_{n+2}^* \cdots I_1^*: H_c^t(U_1 - U_1 \cap F; Q) \to H_c^t(U_{n+2} - U_{n+2} \cap F; Q)$$

is not trivial, for some t. Since $\dim_Q X = n$, it follows that ${}_iE_{n+1}^{s,t} \simeq {}_iE_\infty^{s,t}$; that is, n is the largest possible fibre degree. The image of

$$I_k^r \cdots I_1^r: {}_1E_r^{2s,t} \to {}_{k+1}E_r^{2s,t}$$

contains nonzero elements for each s, whenever $k \leq n + 3 - r$: to see this, we observe that it is true for $r = 2$, and we assume it for $r - 1$. Then for any s, the image of

$$I_{n+3-r} \cdots I_1: {}_1E_{r-1}^{2s,t} \to {}_{n+4-r}E_{r-1}^{2s,t}$$

contains nonzero cocycles which cannot cobound; for if these cocycles cobounded, then upon application of the homomorphism

$$I_{n+4-r}: {}_{n+4-r}E_{r-1}^{2s,t} \to {}_{n+5-r}E_{r-1}^{2s,t},$$

the elements $d_{r-1}(e^{2s+1-r,t+r-2})$ would go into zero, since this image consists of permanent cocycles, and since the homomorphism I_{n+4-r} commutes with d_{r-1}. We had assumed that the homomorphism

$$I_{n+4-r} \cdots I_1: {}_1E_{r-1}^{2s,t} \to {}_{n+5-r}E_{r-1}^{2s,t}$$

is not trivial; thus

$$I_k \cdots I_1 \colon {}_1E_{r-1}^{2s,t} \to {}_{k+1}E_r^{2s,t}$$

has a nontrivial image for all s ($k \leq n + 3 - r$). To complete the proof, let $r = n + 2$; then

$$I_1 \colon {}_1E_{n+2}^{2s,t} \to {}_2E_{n+2}^{2s,t}$$

is nontrivial for all s; thus $H_c^{2s+t}(\hat{U}_2 - \hat{U}_2 \cap F; Q) \neq 0$ for all $s \geq 0$, contrary to the finite-dimensionality of X/S^1. Using Lemma 1.3, we immediately obtain Theorem 3.3.

4. CONCLUSION

We could pursue our methods beyond Theorem 3.3; in particular, it is possible to show, for a transformation group (S^1, X), where X is a locally orientable generalized manifold over the rationals, that $F \subset X$ is also such a manifold. As we said in Section 1, we merely wished to indicate some techniques involving the application of spectral sequences to the study of transformation groups. Borel ([2]) essentially initiated the direction we have followed in this note. In a joint note with Dyer, we shall consider singular fibrations in the sense of Montgomery and Samelson ([8]), and there we shall use another method for studying the global and local cohomology properties of the singular sets.

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