ON MATRICES OF TRACE ZERO

A. A. Albert and Benjamin Muckenhoupt

In 1937, K. Shoda [1] showed that if M is any n-rowed square matrix with elements in a field $\mathfrak{F}$ of characteristic zero, and M has trace $\tau(M) = 0$, then there exist square matrices A and B with elements in $\mathfrak{F}$ such that $M = AB - BA$. Shoda's proof is not valid for a field $\mathfrak{F}$ of characteristic p. The purpose of this note is to furnish a proof holding for any field $\mathfrak{F}$. We begin by deriving the following lemma.

LEMMA. *Let* $M = (m_{ij})$ *be an* n*-rowed square matrix with elements in* $\mathfrak{F}$ *such that*

$$\tau(M) = \sum_{i=1}^{n} m_{ii} = 0, \qquad \sum_{i=1}^{n-1} m_{i,i+1} = 0, \qquad m_{ij} = 0 \text{ for } j \geq i+2.$$

Then $M = AB - BA$, *where* A *and* B *are square matrices with elements in* $\mathfrak{F}$ *and* A *is nonsingular.*

For proof, we let $K = (k_{ij})$ be the n-rowed square matrix with $k_{j+1,j} = 1$ for $j = 1, \cdots, n-1$ and with all other $k_{ij} = 0$. We also let $B = (b_{ij})$ be the matrix with every $b_{i1} = 0$ and $b_{i,i+3} = 0$ for $i = 1, \cdots, n-3$. Then the first row of KB is zero and the $(i-1)$st row of B is the ith row of KB. Also, the $(j+1)$st column of B is the jth column of BK, and the nth column of BK is zero. Then $H = KB - BK = (h_{ij})$, where

(1) $\quad h_{i1} = -b_{i2}, \quad h_{12} = -b_{13}, \quad h_{n-1,n} = b_{n-2,n}, \quad h_{nn} = b_{n-1,n}, \quad h_{ij} = 0 \ (j \geq i+2),$

and

(2) $\quad h_{ij} = b_{i-1,j} - b_{i,j+1} \quad [i = 2, \cdots, n; \ j = 2, \cdots, \min(n-1, i+1)].$

It should now be clear that $m_{ij} = h_{ij} = 0$ for $j \geq i+2$. The other entries h_{ij}, in each column of H except the last, contain a term b_{ij} which does not appear in earlier columns or elsewhere in the same column, and the coefficient of this term is ± 1. It follows that the b_{ij} may be selected successively so that H differs from M in at most two elements, and these are the elements h_{nn} and $h_{n-1,n}$. Since

$$\tau(M) = \tau(H) = \tau(KB - BK) = 0,$$

and $m_{ii} = h_{ii}$ for $i = 1, \cdots, n-1$, it must be clear that we also have $m_{nn} = h_{nn}$. By the form of H we have

Received May 24, 1956.

This paper was sponsored in part by the Office of Ordnance Research, United States Army, under Contract No. DA-11-022-ORD-1571.

$$\sum_{i=1}^{n-1} h_{i,i+1} = -b_{13} + (b_{13} - b_{24}) + \cdots + (b_{n-3,n-1} - b_{n-2,n}) + b_{n-2,n} = 0$$

$$= \sum_{i=1}^{n-1} m_{i,i+1} = 0.$$

Hence $h_{n-1,n} = m_{n-1,n}$, and $H = M$, as desired. Put $A = K + I$, so that $|A| = 1$ and A is nonsingular. Then $AB - BA = (K + I)B - B(K + I) = M$, as desired.

We are now ready to derive our main result.

THEOREM. *Let* M *be an* n*-rowed square matrix with elements in an arbitrary field* $\mathfrak{F}$ *and with* $\tau(M) = 0$. *Then there exist* n*-rowed square matrices* A *and* B *with elements in* $\mathfrak{F}$ *such that* $M = AB - BA$.

We observe first that M is a commutator if and only if any matrix N similar to M is a commutator. Indeed, if $N = P^{-1}MP = AB - BA$, then

$$M = (PAP^{-1})(PBP^{-1}) - (PBP^{-1})(PAP^{-1}).$$

Hence we may assume that M is in rational canonical form, that is,

$$M = \operatorname{diag}\{C_{\phi_1}, \cdots, C_{\phi_k}\},$$

where the $\phi_i = \phi_i(x)$ are the nontrivial invariant factors of $xI - M$, where $\phi_i(x)$ divides $\phi_{i-1}(x)$ for $i = 2, \cdots, k$, and where C_ϕ is the companion matrix

$$(3) \qquad \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_m & \alpha_{m-1} & \alpha_{m-2} & \cdots & \alpha_1 \end{pmatrix}$$

of the polynomial $\phi = \phi(x) = x^m - (\alpha_1 x^{m-1} + \cdots + \alpha_m)$. Then the matrix M has elements 1 and 0 above the diagonal. Consequently, there exists a similarity transformation by means of which the elements 1 may be replaced(1) by a sequence $1, -1, 1, -1, \cdots$. Indeed, we may multiply the third row and column in (3) by -1, if necessary, and replace the second 1 by -1. Assume then that we have carried out the similarity transformation which makes the first k nonzero elements $m_{i,i+1}$ *alternate* in sign. Then the $(k+1)$st element occurs in the sth row and $(s+1)$st column and can be changed in sign, if necessary, by the similarity transformation which merely multiplies the $(s+1)$st row and column by -1.

The argument just given shows that, by passing to a similar matrix if necessary, we may assume that $M = (m_{ij})$, where $m_{ij} = 0$ for $j \geq i + 2$, where $\tau(M) = 0$, and

(1) We can actually replace the elements $m_{i,i+1}$ by products $d_i m_{i+1} d_i^{-1} = \mu_{i,i+1}$, by means of a diagonal similarity transformation. The d_i can clearly be selected so that $\Sigma_{i-1}^{n-1} \mu_{i,i+1} = 0$ except when there is only one nonzero $m_{i,i+1}$ or when $\mathfrak{F}$ is the field of two elements and there is an odd number of $m_{i,i+1} \neq 0$.

where $m_{i,i+1} = 1, -1,$ or 0 and the nonzero $m_{i,i+1}$ alternate in sign.

If there are an even number of nonzero $m_{i,i+1}$, we have the property

$$\sum_{i=1}^{n-1} m_{i,i+1} = 0 \quad .$$

of the lemma, and $M = AB - BA$ as desired. If the number is odd, the matrix M has the form

$$M = \begin{pmatrix} 0 & u \\ v' & M_1 \end{pmatrix},$$

where u and v are $1 \times (n - 1)$ matrices and M_1 is an $(n - 1)$-rowed square matrix. But then M_1 has all of the properties of our lemma, and therefore $M_1 = A_1B_1 - B_1A_1$, for some nonsingular matrix A_1. Take

$$A = \begin{pmatrix} 0 & 0 \\ 0 & A_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -uA_1^{-1} \\ A_1^{-1}v' & B_1 \end{pmatrix},$$

and see that

$$AB - BA = \begin{pmatrix} 0 & 0 \\ v' & A_1B_1 \end{pmatrix} - \begin{pmatrix} 0 & -u \\ 0 & B_1A_1 \end{pmatrix} = M,$$

as desired.

REFERENCE

1. K. Shoda, *Einige Sätze über Matrizen*, Jap. J. Math., 13 (1936), 361-365.

The University of Chicago