THE VIETORIS MAPPING THEOREM FOR BICOMPACT SPACES II

E. G. Begle

In an earlier paper with this title [1] we extended to the case of arbitrary bicompact spaces the theorem of Vietoris to the effect that if the inverse image of each point of an image space is homologically trivial in all dimensions up to and including n, then the homomorphism of the n-dimensional homology group of the original space into that of the image space is actually an isomorphism onto. However, we did not consider there the action of the mapping on the homology groups of dimension $n + 1$. The following theorem is devoted to this case.

THEOREM. *If the coefficient group is an elementary compact group or is a field, and if* f *is a Vietoris mapping of order* n *of* X *onto* Y, *then the homomorphism of* $H_V^{n+1}(X)$ *into* $H_V^{n+1}(Y)$ *induced by* f *is a homomorphism onto.*

Proof. We must show that for each $(n + 1)$-V-cycle Z of Y, there exists an $(n + 1)$-V-cycle Γ of X such that $f\Gamma \sim Z$. We first recall that if Δ is a Čech cycle of X, then there is associated with it a V-cycle Γ defined as follows: for each covering M of X, let M' be a star-refinement of M. Define $\Gamma(M) = \phi(\Delta(M))$, where ϕ is the simplicial mapping, defined in Section 2 of [1], of the nerve $\overline{M}'$ of M' into X(M). We use the notation $\Gamma = \phi(\Delta)$.

We now assign to each coordinate Z(N) of Z a Čech cycle Δ_N of X with the property that $f\Gamma_N(M) \sim Z(N)$ in Y(N), where M is the covering $f^{-1}(N)$ and $\Gamma_N = \phi(\Delta_N)$. To do this, let P be any covering of X such that $P <^* M$. Let Q be a normal refinement of P [3, p. 678 and 2, p. 216]. Let L be a star-refinement of Q. Now, set $\gamma_N = T(Z(R))$, where $R = R(L, N)$ and T are given by Lemma 2 of [1]. Let ζ be the simplicial mapping, defined in Section 2 of [1], of X(L) into $\overline{Q}$. Then $\zeta(\gamma_N)$ is a cycle on $\overline{Q}$, and since Q is a normal refinement of P, $\pi_Q^P\zeta(\gamma_N)$ is the coordinate on $\overline{P}$ of some Čech cycle Δ_N.

To see that Δ_N has the required property, we first note that, by Lemma 2 of [1], $f(\gamma_N) \sim Z(R)$ in Y(N). But $Z(R) \sim Z(N)$ in Y(N), so $f(\gamma_N) \sim Z(N)$ in Y(N). Also, as in Section 2 of [1], $\gamma_N \sim \phi\pi_Q^P\zeta(\gamma_N)$ in X(M). But $\pi_Q^P\zeta(\gamma_N) = \Delta_N(P)$, so

$$\phi\pi_Q^P\zeta(\gamma_N) = \Gamma_N(M).$$

Hence $f(\Gamma_N(M)) \sim Z(N)$ in Y(N).

Now for each covering N of Y let J_N be the collection of all Čech cycles Δ of X having the property that $f\Gamma(M) \sim Z(N)$ in Y(N), where $M = f^{-1}(N)$ and $\Gamma = \phi(\Delta)$. We have just shown that J_N is not vacuous. It is clear that J_N is closed in the usual topology assigned to the group of Čech cycles of X [2, p. 215]. We shall show that the collection $\{J_N\}$ has the finite intersection property.

Let $N_1, \cdots, N_m$ be coverings of Y. Choose $O_i <^* N_i$ and let $M_i = f^{-1}(N_i)$, $P_i = f^{-1}(O_i)$. Let N_0 be a common refinement of $O_1, \cdots, O_m$. Let $M_0 = f^{-1}(N_0)$ and let $P_0 = {}^*M_0$. Now construct Δ_{N_0}, using this P_0. Then $f\phi_0(\Delta_{N_0}(P_0)) \sim Z(N_0)$ in $Y(N_0)$, where ϕ_0 is the simplicial mapping of $\overline{P}_0$ into $X(M_0)$. Now consider a fixed i $(1 \leq i \leq m)$. Since Z is a V-cycle, $Z(N_0) \sim Z(N_i)$ in $Y(N_i)$. Since Δ_{N_0} is a Čech

Received February 18, 1956.

cycle, $\pi(\Delta_{N_0}(P_0)) \sim \Delta_{N_0}(P_i)$ on $\overline{P}_i$, where π is the projection from $\overline{P}_0$ to $\overline{P}_i$. Then

$$\phi_i\pi(\Delta_{N_0}(P_0)) \sim \phi_i(\Delta_{N_0}(P_i))$$

in $X(M_i)$, where ϕ_i is the simplicial mapping of $\overline{P}_i$ into $X(M_i)$. But, as in Section 2 of [1],

$$\phi_0(\Delta_{N_0}(P_0)) \sim \phi_i\pi(\Delta_{N_0}(P_0))$$

in $X(M_i)$. Hence

$$f\phi_i(\Delta_{N_0}(P_i)) \sim f\phi_0(\Delta_{N_0}(P_0))$$

in $Z(N_i)$. But this means that Δ_{N_0} is in J_{N_i}.

Since the group of $(n+1)$-dimensional Čech cycles of X is compact, the intersection of all J_N is not vacuous. Let Δ be any element in this intersection. Then $\Gamma = \phi(\Delta)$ is such that $f\Gamma(M) \sim Z(N)$ in $Y(N)$ for each covering N of Y, and so $f\Gamma \sim Z$. This completes the proof.

The following example shows that neither the theorem above nor Theorem 2 of [1] is correct when the coefficient group is taken to be the group of integers. Let S be a solenoid. Using the original Vietoris representation of S [4, p. 459], let p and q be points in the same cross-section of S with coordinates 0.000··· and 0.2020202··· respectively. Let I be an arc with end points p and q and with no other points in common with S. Let $X = S \cup I$. Let f be the mapping which identifies all points of S and is the identity in the interior of I. Then $Y = f(X)$ is a circle.

An easy calculation shows that neither S nor X carries a nontrivial 1-dimensional V-cycle, while Y does. Since S is connected, f is a Vietoris mapping of order 0, which shows that the theorem above fails for this coefficient group. Also, f is such that, for each y in Y, both the 0-dimensional and the 1-dimensional homology groups of $f^{-1}(y)$ are trivial; therefore Theorem 2 of [1] also fails for this coefficient group.

REFERENCES

1. E. G. Begle, *The Vietoris mapping theorem for bicompact spaces*, Ann. of Math. (2) 51 (1950), 534-543.

2. S. Lefschetz, *Algebraic topology*, Amer. Math. Soc. Colloquium Publications 27 (1942), New York.

3. N. E. Steenrod, *Universal homology groups*, Amer. J. Math. 58 (1936), 661-701.

4. L. Vietoris, *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen*, Math. Ann. 97 (1927), 454-472.

Yale University