DISTRIBUTION OF EIGENVALUES OF CERTAIN INTEGRAL OPERATORS

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1. INTRODUCTION

The classical theorem of H. Weyl concerning the asymptotic behavior of the eigenvalues of the Laplacian can be stated (in three-dimensional space, say) as follows.

Consider the integral equation

$$\frac{1}{2\pi}\int_{\Omega}\frac{\phi(\vec{\rho})}{|\vec{\rho}-\vec{r}|}d\vec{\rho}=\lambda\phi(\vec{r}),\quad \vec{r}\in\Omega, \tag{1.1}$$

where Ω *is a region. Then*

$$1/\lambda_n \sim \left(\frac{3\pi^2}{\sqrt{2}|\Omega|}\right)^{2/3} n^{2/3}, \quad \text{as } n\to\infty, \tag{1.2}$$

where $|\Omega|$ *denotes the volume of* Ω. In a previous paper [1] a proof of this theorem, based on the theory of Brownian motion (Wiener measure), was sketched.

It is the purpose of this paper to prove an analogous theorem for integral equations of the form

$$\int_{\Omega}\frac{\phi(\vec{\rho})}{|\vec{r}-\vec{\rho}|^{\alpha/2}}d\vec{\rho}=\lambda\phi(\vec{r})\quad (0<\alpha\leq 2). \tag{1.3}$$

Unlike in the case (1.1), there is no equivalent formulation in terms of a differential equation. The method of proof will be illustrated on the one-dimensional case, and to obtain a somewhat more general result we shall consider the integral equation

$$\int_{-a}^{a}\frac{\phi(y)V(y)}{|y-x|^{\alpha}}dy=\lambda\phi(x), \tag{1.4}$$

where $V(y)$ is a continuous function bounded away from 0, that is,

$$V(y)>m>0,\quad (-a\leq y\leq a). \tag{1.5}$$

To avoid complications of a minor nature we shall consider (1.4) only for $\alpha<1/2$, indicating later how this restriction can be removed. The final result is given by formula (4.3). The proof will be an adaptation of the argument used in §11 of [1].

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2. PRELIMINARIES

Let $x(\tau)$ $(x(0) = 0)$ be the stable process of exponent $\beta = 1 - \alpha$, and let $\chi_A(x)$ be the characteristic function of the set $A \subset (-a, a)$. We consider the integral

$$(2.1) \qquad \int_0^\infty E\left\{ e^{-u\int_0^t V(x + x(\tau))d\tau} \chi_A(x + x(t))\right\} dt$$

for $u \geq 0$.

Following the derivation of §11 of [1], we get

$$(2.2) \quad \nu_k = \int_0^\infty E\left\{\left(\int_0^t V(x + x(\tau))d\tau\right)^k \chi_A(x + x(t))\right\}dt$$

$$= k!D^{k+1}(\beta)\int_A dx_{k+1} \int_{-a}^{a}\cdots\int \frac{V(x_1)V(x_2)\cdots V(x_k)}{|x - x_1|^\alpha |x_2 - x_1|^\alpha \cdots |x_{k+1} - x_k|^\alpha} dx_1 \cdots dx_k,$$

where

$$(2.3) \qquad D(\beta) = \frac{1}{\pi}\int_0^\infty \frac{\cos \eta}{\eta^\beta} d\eta .$$

Let $\mu_1, \mu_2, \cdots$ be the eigenvalues and $\psi_1(x), \psi_2(x), \cdots$ the corresponding normalized eigenfunctions of the integral equation

$$(2.4) \qquad D(\beta)\int_{-a}^{a} \frac{\sqrt{V(x)}\sqrt{V(y)}}{|x - y|^\alpha} \psi(y)\, dy = \mu\, \psi(x).$$

We get

$$(2.5) \qquad \nu_k = k!\sum_{j=1}^{\infty} \mu_j^{k+1} \int_A \frac{\psi_j(y)}{\sqrt{V(y)}} dy \frac{\psi_j(x)}{\sqrt{V(x)}} ,$$

and consequently

$$(2.6) \quad \int_0^\infty E\left\{ e^{-u\int_0^t V(x + x(\tau))d\tau} \chi_A(x + x(t))\right\}dt = \sum_{j=1}^{\infty} \frac{\mu_j}{1 + u\mu_j}\int_A \frac{\psi_j(y)}{\sqrt{V(y)}} dy \frac{\psi_j(x)}{\sqrt{V(x)}} .$$

The passage from (2.5) to (2.6) requires a word of justification.

For sufficiently small u the justification is clear. Considering u as a complex variable with $\Re u > 0$, we note that both sides of (2.6) are analytic functions of u.

Since they agree for sufficiently small positive u, they agree for all u with $\Re u > 0$ and hence for all positive u.

Let now

$$\sigma_A(\gamma;\, t) = \text{Prob}\left\{ \int_0^t V(x + x(\tau))d\tau < \gamma,\ x + x(t) \in A \right\}. \tag{2.7}$$

The left side of (2.6) can be rewritten in the equivalent form

$$\int_0^\infty \int_0^\infty e^{-u\gamma} d\gamma\, \sigma(\gamma;\, t)dt = \int_0^\infty e^{-u\gamma}\, d\left\{ \int_0^\infty \sigma(\gamma;\, t)dt \right\},$$

while the right-hand side is clearly

$$\int_0^\infty e^{-u\gamma}\, d\left\{ -\sum_{j=1}^\infty \mu_j\, e^{-\gamma/\mu_j} \int_A \frac{\psi_j(y)}{\sqrt{V(y)}} dy\, \frac{\psi_j(x)}{\sqrt{V(x)}} \right\}.$$

Thus

$$\int_0^\infty \sigma(\gamma;\, t)dt = C - \sum_{j=1}^\infty \mu_j e^{-\gamma/\mu_j} \int_A \frac{\psi_j(y)}{\sqrt{V(y)}} dy\, \frac{\psi_j(x)}{\sqrt{V(x)}},$$

and clearly

$$C = \int_0^\infty \sigma(\infty;\, t)dt.$$

Finally,

$$\int_0^\infty [\sigma(\infty;\, t) - \sigma(\gamma;\, t)]dt = \sum_{j=1}^\infty \mu_j\, e^{-\gamma/\mu_j} \int_A \frac{\psi_j(y)}{\sqrt{V(y)}} dy\, \frac{\psi_j(x)}{\sqrt{V(x)}},$$

or

$$\begin{aligned} &\int_0^\infty \text{Prob}\left\{ \int_0^t V(x + x(\tau))\, d\tau \geq \gamma,\ x + x(t) \in A \right\} dt \\ &\quad = \sum_{j=1}^\infty \mu_j\, e^{-\gamma/\mu_j} \int_A \frac{\psi_j(y)}{\sqrt{V(y)}} dy\, \frac{\psi_j(x)}{\sqrt{V(x)}}\,. \end{aligned} \tag{2.8}$$

Some of the steps above were purely formal, but the justification is quite easy.

3. PASSAGE TO CONDITIONAL PROBABILITIES

Let now A be the interval $(x - \varepsilon, x + \varepsilon)$. The condition $x + x(t) \in A$ becomes now $-\varepsilon < x(t) < \varepsilon$, and we can rewrite (2.8) in the equivalent form

$$\int_0^\infty \frac{\text{Prob}\{-\varepsilon < x(t) < \varepsilon\}}{2\varepsilon} \frac{\text{Prob}\left\{\int_0^t V(x + x(\tau))d\tau \geq \gamma,\ -\varepsilon < x(t) < \varepsilon\right\}}{\text{Prob}\{-\varepsilon < x(t) < \varepsilon\}} dt \tag{3.1}$$

$$= \sum_{j=1}^{\infty} \mu_j e^{-\gamma/\mu_j} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \frac{\psi_j(y)}{\sqrt{V(y)}} dy \frac{\psi_j(x)}{\sqrt{V(x)}} .$$

We must now show that

$$\lim_{\varepsilon \to 0} \frac{\text{Prob}\left\{\int_0^t V(x + x(\tau))d\tau \geq \gamma,\ -\varepsilon < x(t) < \varepsilon\right\}}{\text{Prob}\{-\varepsilon < x(t) < \varepsilon\}} \tag{3.2}$$

exists.

Let

$$F_\varepsilon(\gamma) = \frac{\text{Prob}\left\{\int_0^t V(x + x(\tau))d\tau < \gamma,\ -\varepsilon < x(t) < \varepsilon\right\}}{\text{Prob}\{-\varepsilon < x(t) < \varepsilon\}}.$$

It is easily shown that

$$\lim_{\varepsilon \to 0} \int_0^\infty \gamma^k dF_\varepsilon(\gamma) = \lim_{\varepsilon \to 0} \frac{E\left\{\left(\int_0^t V(x + x(\tau))d\tau\right)^k,\ -\varepsilon < x(t) < \varepsilon\right\}}{\text{Prob}\{-\varepsilon < x(t) < \varepsilon\}}$$

exists and is clearly less than $(Mt)^k$, where $M = \sup V(x)$ $(-a \leq x \leq a)$. It thus follows that $\lim_{\varepsilon \to 0} F_\varepsilon(\gamma)$ exists (in the usual sense of convergence of distribution functions) and hence limit (3.2) also exists. It is natural to set

$$\lim_{\varepsilon \to 0} (1 - F_\varepsilon(\gamma)) = \text{Prob}\left\{\int_0^t V(x + x(\tau))d\tau \geq \gamma \mid x(t) = 0\right\}. \tag{3.3}$$

Since the kernel

$$\frac{\sqrt{V(x)}\sqrt{V(y)}}{|x - y|^\alpha} \in L^2 \quad (-a \leq x \leq a,\ -a \leq y \leq a)$$

and its first iterate is clearly continuous we have, by Mercer's theorem, that

$$\sum_1^\infty \mu_j^2 \psi_j(x)\psi_j(y)$$

converges absolutely. Since

$$e^{-\gamma/\mu_j} < A\mu_j,$$

it follows that

$$\sum_1^\infty \mu_j e^{-\gamma/\mu_j} \psi_j(x)\psi_j(y)$$

converges absolutely. Hence (3.1) can be written in the form

$$C(\beta)\int_0^\infty t^{-1/\beta}\ \text{Prob}\left\{\int_0^t V(x+x(\tau))d\tau \geq \gamma \mid x(t)=0\right\}dt$$

(3.4)

$$= \frac{1}{V(x)}\sum_1^\infty \mu_j e^{-\gamma/\mu_j}\psi_j^2(x);$$

for

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\ \text{Prob}\{-\varepsilon < x(t) < \varepsilon\}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}dx\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-t|\xi|^\beta}e^{i\xi x}d\xi\right\} = C(\beta)t^{-1/\beta},$$

with

$$C(\beta) = \frac{1}{\pi}\int_0^\infty e^{-\xi^\beta}d\xi . \tag{3.5}$$

Should the reader become disturbed by the fact that $t^{-1/\beta}$ is not integrable near $t = 0$, he should notice that for

$$t < \gamma/M \qquad (M = \sup V(x))$$

we have

$$\text{Prob}\left\{\int_0^t V(x+x(\tau))d\tau \geq \gamma \mid x(t)=0\right\} = 0,$$

and hence the integration in (3.4) does not actually extend to 0. A simple change of variable transforms (3.4) into the more convenient form

$$C(\beta)\gamma^{1-1/\beta}\int_0^\infty t^{-1/\beta}\,\mathrm{Prob}\left\{\int_0^t V(x+x(\gamma\tau))d\tau \geq 1 \mid x(\gamma t)=0\right\}dt$$

$$=\frac{1}{V(x)}\sum_1^\infty \mu_j\, e^{-\gamma/\mu_j}\,\psi_j^2(x),$$

which implies immediately

$$\sum_1^\infty \mu_j e^{-\gamma/\mu_j}$$

(3.6)

$$=C(\beta)\gamma^{1-1/\beta}\int_0^\infty t^{-1/\beta}dt\int_{-a}^{a} V(x)\,\mathrm{Prob}\left\{\int_0^t V(x+x(\gamma t))d\tau \geq 1 \mid x(\gamma t)=0\right\}dx.$$

Intuitively,

$$\lim_{\gamma\to 0}\mathrm{Prob}\left\{\int_0^t V(x+x(\gamma\tau))d\tau \geq 1 \mid x(\gamma t)=0\right\}=\begin{cases}1 & \text{if } tV(x)\geq 1,\\ 0 & \text{if } tV(x)<1.\end{cases} \tag{3.7}$$

It is (3.7) and similar formulas which are the heart of the probabilistic approach to the problem of distribution of eigenvalues. For the Wiener process (Brownian motion) the analogue of (3.7) is what I called in [1] the "principle of not feeling the boundary" and which has been most thoroughly justified and employed by D. Ray in [2]. For stable processes the proof of (3.7) is still very simple, because of the strong assumptions we imposed on $V(x)$. In fact,

$$E\left\{\int_0^t V(x+x(\gamma\tau))d\tau \mid x(\gamma t)=0\right\}$$

$$=\frac{\int_0^t\int_{-\infty}^{\infty} V(x+\eta)P(\eta;\gamma\tau)P(\eta;\gamma(t-\tau))\,d\eta\,d\tau}{P(0;\gamma t)},$$

where

$$P(y;t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi y}e^{-t|\xi|^\beta}d\xi,$$

and it follows almost immediately that

$$\lim_{\gamma\to 0}E\left\{\int_0^t V(x+x(\gamma\tau))d\tau \mid x(\gamma t)=0\right\}=tV(x). \tag{3.8}$$

Similarly it follows that

(3.9) $$\lim_{\gamma\to 0} E\left\{\left(\int_0^t V(x + x(\gamma\tau))d\tau\right)^2 \mid x(\gamma t) = 0\right\} = t^2 V^2(x) .$$

Now, clearly (3.8) and (3.9) imply (3.7).

4. CONCLUSION OF THE PROOF

From (3.6) and (3.7) we obtain that

(4.1) $$\sum_1^\infty \mu_j e^{-\gamma/\mu_j} \sim C(\beta)\gamma^{1-1/\beta} \int_{-a}^a V(x)\left(\int_{1/V(x)}^\infty t^{-1/\beta}dt\right)dx$$

$$= \frac{C(\beta)\int_{-a}^a V^{1/\beta}(x)\,dx}{1/\beta - 1} \cdot \gamma^{1-1/\beta}$$

as $\gamma\to 0$, and hence by Karamata's Tauberian theorem

(4.2) $$\sum_{1/\mu_j<\mu} \mu_j \sim \frac{C(\beta)\int_{-a}^a V^{1/\beta}(x)\,dx}{(1/\beta - 1)\ \Gamma(1/\beta)}\ \mu^{1/\beta-1} \qquad (\mu\to\infty).$$

Since

$$C(\beta) = \frac{1}{\pi\beta}\,\Gamma(1/\beta),$$

formula (4.2) can be written in the simpler form

$$\sum_{1/\mu_j<\mu} \mu_j \sim \frac{\frac{1}{\pi}\int_{-a}^a V^{1/\beta}(x)\,dx}{1 - \beta}\ \mu^{1/\beta-1} ,$$

and since the μ_j's form a decreasing sequence it follows, in the same way as at the end of §11 of [1], that

$$\sum_{1/\mu_j<\mu} 1 \sim \frac{1}{\pi}\int_{-a}^a V^{1/\beta}(x)\,dx\ \mu^{1/\beta} ,$$

or

(4.3) $$\frac{1}{\mu_n} \sim \frac{n^\beta}{\left(\frac{1}{\pi}\int_{-a}^a V^{1/\beta}(x)\,dx\right)^\beta} .$$

To remove the restriction $\alpha < 1/2$, we note that if $1/2 \leq \alpha < 1$, a sufficiently high iterate of the kernel of (2.4) is in L^2. The proof can now be carried out by considering instead of (2.1) the expression

$$\int_0^\infty E\left\{ e^{-u\left(\int_0^t V(x+x(\tau))d\tau\right)^\ell} \chi_A(x + x(t))\right\} dt ,$$

where ℓ (depending on α) is sufficiently large so that the corresponding kernel is L^2.

In conclusion it might be mentioned that Pólya and Szegö [3; p. 29, formula 19] proved that

$$(4.4)\quad D(\beta)\int_{-1}^{1} (1 - x^2)^{-\beta/2} \frac{P_n^{(\alpha/2)}(x)}{|x - y|^\alpha} dx = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)} P_n^{(\alpha/2)}(y) \quad (-1 \leq y \leq 1) ,$$

where the P's are Jacobi polynomials. Thus for

$$V(x) = (1 - x^2)^{-\beta/2} , \; a = 1 ,$$

we have

$$\mu_n = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)} ,$$

and it is easily checked that (4.3) holds. Our proof is not strictly applicable to this case, because here $V(x)$ is not bounded from above. However the proof could be modified to include such cases provided, of course, $V(x)$ does not become infinite too strongly.

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