A NOTE ON POWER SERIES WHICH DIVERGE EVERYWHERE ON THE UNIT CIRCLE

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It is well known that there exist power series $\Sigma a_n z^n$ with $\lim a_n = 0$ which diverge for all points z on the unit circle. An example of this kind was constructed by N. Lusin (see [3, pp. 69-71]). The coefficients in Lusin's example have arguments that are everywhere dense in the interval $(0, 2\pi)$, and the question arises as to whether this distribution of the coefficients in the complex plane constitutes an intrinsic property of power series of the type mentioned above. This is by no means the case, as is shown in Theorem 1 below, where the coefficients a_n are all real and nonnegative.

In Theorem 2, we apply the so-called 'doubling process', first used by Erdös, Piranian and the author (see [1, § 3]), in order to construct a cosine series $\Sigma a_n \cos n\theta$, with $a_n \geq 0$ and $\lim a_n = 0$, which diverges for all real values of θ.

THEOREM 1. *There exists a power series* $\Sigma_{n=0}^{\infty} a_n z^n$, *with* $a_n \geq 0$ *and* $\lim a_n = 0$, *which diverges at all points of the unit circle.*

Let

$$F(z) = \sum_{m=1}^{\infty} z^{k_m} S_m(z)/m, \tag{1}$$

where

$$S_m(z) = 1 + z^m + z^{2m} + \cdots + z^{(m-1)m}. \tag{2}$$

The exponents k_m in (1) are to be positive integers large enough so that the powers of z in the $(m+1)$st term of the series are higher than those in the mth term.

Let $e^{i\theta}$ be an arbitrary point on the unit circle, and let h/m be one of the infinitely many rational approximations of $\theta/2\pi$ for which $|\theta/2\pi - h/m| < 1/2m^2$. (See, for instance, [4, p. 48, Theorem 14].) If we put $\theta = (2\pi h + \beta)/m$, then $|\beta| < \pi/m$ and, by (2),

$$|S_m(e^{i\theta})| = \left|\sum_{\mu=0}^{m-1} e^{i\mu\beta}\right| > 2m/\pi.$$

(For a proof of this inequality see, for instance, [2, p. 530, Lemma B].) Thus

$$|(e^{i\theta})^{k_m} S_m(e^{i\theta})/m| > 2/\pi$$

for infinitely many m, and the Taylor series of $F(z)$ cannot converge at the point $z = e^{i\theta}$.

Received by the editors December 23, 1954.

THEOREM 2. *There exists a cosine series* $\Sigma_{n=0}^{\infty} a_n \cos n\theta$, *with* $a_n \geq 0$ *and* $\lim a_n = 0$, *which diverges for all real values of* θ.

Let

$$(3) \qquad G(z) = \sum_{m=1}^{\infty} \left(z^{k_m} T_m(z)/m + z^{2k_m} T_m(z)/m \right),$$

where

$$(4) \qquad T_m(z) = 1 + z^m + z^{2m} + \cdots + z^{[m/12]m}.$$

The exponents k_m in (3) are to be positive integers large enough so that the powers of z in the second 'half' of each term of the series are higher than those in the first 'half' of that term, and so that the powers of z in the $(m+1)$st term of the series are higher than those in the mth term. It then suffices to show that the real part of the series on the right-hand side of (3) does not converge at $z = e^{i\theta}$, where $e^{i\theta}$ is an arbitrary point on the unit circle.

Let h, m and β have the same meaning as in the proof of Theorem 1. By (4), we have

$$(5) \qquad T_m(e^{i\theta}) = \sum_{\mu=0}^{[m/12]} e^{i\mu\beta}.$$

Since $|\mu\beta| \leq [m/12]|\beta| < \pi/12$, for $\mu = 0, 1, 2, \cdots, [m/12]$, we obtain from (5)

$$|T_m(e^{i\theta})| \geq \Re\, T_m(e^{i\theta}) > ([m/12] + 1)\cos(\pi/12) > Km,$$

where K is a positive universal constant, and

$$|\arg T_m(e^{i\theta})| < \pi/12.$$

Hence, at least one of the two quantities

$$(e^{i\theta})^{k_m} T_m(e^{i\theta}) \quad \text{and} \quad (e^{i\theta})^{2k_m} T_m(e^{i\theta})$$

lies in the union of the two sectors

$$|w| > Km, \quad |\arg w| < 5\pi/12,$$

and

$$|w| > Km, \quad |\arg w - \pi| < 5\pi/12$$

of the w-plane and, consequently, has a real part whose absolute value is greater than Lm, where L is again a positive universal constant. This shows that, for $z = e^{i\theta}$ and for infinitely many m, the real part of at least one of the two 'halves' of the mth term of the series on the right-hand side of (3) exceeds L in modulus, and the proof of Theorem 2 is concluded.

REFERENCES

1. P. Erdös, F. Herzog and G. Piranian, *Sets of divergence of Taylor series and of trigonometric series*, Math. Scand. (to appear).

2. F. Herzog and G. Piranian, *Sets of convergence of Taylor series I*, Duke Math. J. 16 (1949), 529-534.

3. E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, 2nd edition, Berlin, 1929.

4. O. Perron, *Die Lehre von den Kettenbrüchen*, Leipzig-Berlin, 1929.

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