THE CONTENT OF A YOUNG DIAGRAM

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1. INTRODUCTION. As is well known, the substitutional analysis of Alfred Young leads from the diagram $[\lambda]$ to an explicit representation theory of S_n. The purpose of this paper is to throw light on the *graph* $G[\lambda]$ of $[\lambda]$, discussed elsewhere [4, Part III; 5], and on the significance in this context of Frobenius' notation for a partition. It will be helpful to exhibit the ideas involved in a highly intuitive form, and this we attempt to do here.

Consider a doubly infinite matrix $G = (g_{ij})$, where

$$g_{ij} = j - i \quad (i, j = -\infty, \cdots, -1, 0, 1, \cdots, +\infty),$$

and imagine a given $[\lambda]$ superimposed upon G so that the (i, j) node of $[\lambda]$ covers g_{ij}. The operators T and S defined in (2.4) correspond to the horizontal and vertical displacement of $[\lambda]$ over G. The *content* $C[\lambda]$ of $[\lambda]$ corresponds to the set of elements of G covered by $[\lambda]$. In (3.9) we obtain a necessary and sufficient condition that a given content should be admissible, i.e. should correspond to a Young diagram $[\lambda]$, and in §4 we show how to construct $[\lambda]$ when its content is given.

In §5 we pass to the modular theory by replacing the g_{ij} by their nonnegative residues modulo q. The operators T and S are now periodic of order q, so far as the content is concerned. This periodicity shows itself in the fixed content of a q-hook under T and S, which corresponds to a complete set of residues modulo q. This leads in §6 to a criterion that a diagram $[\lambda]$ be a q-core in terms of Frobenius' notation, the criterion having already been given in Young's case [6].

The paper concludes with the adaptation of the familiar partition generating function [2] to yield the content $C[\lambda]$ for all $[\lambda]$.

2. THE CONCEPT OF CONTENT. As a tool in the study of Young diagrams we introduce some concepts associated with the lattice of integer points in the plane. For this theory it is customary to alter the usual coordinate system so that the positive direction is downward on the first axis, towards the right on the second axis. To proceed more formally, let I be the set of integers $\{\cdots, -2, -1, 0, 1, 2, \cdots\}$. We give the name *node* to any element (i, j) of the cartesian product $I \times I$. Let $\{x_i \mid i \in I\}$ be a set of commutative indeterminates. We call x_{j-i} the *content of the node* (i,j) and write

$$x_{j-i} = C\{(i, j)\}. \tag{2.1}$$

If $M = \{(i_1, j_1), \cdots, (i_n, j_n)\}$ is a finite subset of $I \times I$ we define the *content* $C(M)$ of M to be the product of the contents of its nodes, that is,

$$C(M) = \prod_{h=1}^{n} C\{(i_h, j_h)\}. \tag{2.2}$$

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We usually collect terms and write the monomial C(M) in the form

$$C(M) = \prod_{h=-\infty}^{\infty} x_h^{\mu_h}. \tag{2.3}$$

The set of nodes (i, j) in I×I for which $j - i = h$ we call the h-th *diagonal*. Clearly μ_h is the number of nodes in M which lie on the h-th diagonal.

We consider two operators on I×I. These are T, a one-unit shift to the right, and S, a one-unit shift downward, i.e.

$$(i, j)T = (i, j + 1), \quad (i, j)S = (i + 1, j). \tag{2.4}$$

We apply T and S to sets by shifting each node of the set. We also define T and S as operators on monomials by

$$(\Pi x_h^{\mu_h})T = \Pi x_{h+1}^{\mu_h}, \quad (\Pi x_h^{\mu_h})S = \Pi x_{h-1}^{\mu_h}. \tag{2.5}$$

Note that we have defined T and S as right operators; they have no effect from the left. For example,

$$Tx_0Tx_0Tx_0Tx_0 = T^4x_3x_2x_1x_0 = x_3x_2x_1x_0.$$

We shall use the notation $(Tx_0)^4$ for the extreme left-hand member of this equation; and, in general, we shall understand that powers of expressions involving operators are to be written out before the operators are applied.

It follows readily from definitions (2.4) and (2.5) that, for each finite set M,

$$C(MT) = (C(M))T, \quad C(MS) = (C(M))S \tag{2.6}$$

and

$$C(MTS) = C(MST) = C(M). \tag{2.7}$$

3. THE CONTENT OF A YOUNG DIAGRAM. Corresponding to each partition (λ) of n,

$$\lambda_1 + \cdots + \lambda_k = n \quad (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0), \tag{3.1}$$

we have the Young diagram [λ], consisting of the n nodes

$$(1, 1), \cdots, (1, \lambda_1), (2, 1), \cdots, (2, \lambda_2), \cdots, (k, 1), \cdots, (k, \lambda_k).$$

The diagram [λ] is completely determined by its set of first column hook lengths

$$l_i = \lambda_i + k - i \quad (i = 1, \cdots, k) \tag{3.2}$$

where, of course,

$$l_1 > l_2 > \cdots > l_k > 0. \tag{3.3}$$

Conversely, corresponding to each set of strictly decreasing positive integers there exists a uniquely determined Young diagram $[\lambda]$ which has these numbers as its set of first column hook lengths.

We now study the content of a Young diagram $[\lambda]$:

$$C[\lambda] = \prod_{h=-\infty}^{\infty} x_h^{\mu_h} = \prod_{h=-k+1}^{\lambda_1 - 1} x_h^{\mu_h}. \tag{3.4}$$

The second equality follows from the fact that for each node (i, j) of $[\lambda]$ we have $1 \le i \le k$ and $1 \le j \le \lambda_j$, whence (i, j) lies in the h-th diagonal, where $-k < h < \lambda_1$. Moreover, the μ_h nodes lying in the h-th diagonal are

$$\{(1, h+1), \cdots, (\mu_h, h+\mu_h)\} \text{ if } h \ge 0$$

and

$$\{(-h+1, 1), \cdots, (-h+\mu_h, \mu_h)\} \text{ if } h < 0.$$

It follows that

(3.5) *Two Young diagrams* $[\lambda]$ *and* $[\lambda']$ *are identical if and only if their contents are equal.*

We now determine which monomials are contents of Young diagrams, and we give two methods for determining a partition from the content of its Young diagram.

Let (λ) be a partition of n in the form (3.1), and let h_{ij} denote the length of the hook H_{ij} whose corner is the node (i, j) of $[\lambda]$. The hooks H_{11}, H_{22}, $\cdots$ are called *diagonal hooks*. The number r of diagonal hooks of $[\lambda]$ is called the *rank* of (λ). Clearly $r = \mu_0$. Since each node of $[\lambda]$ lies in exactly one diagonal hook, the positive integers $h_{11}, \cdots, h_{rr}$ form a partition of n. However, since several partitions (λ) may have the same h_{ii}, we go one step further and write

$$h_{ii} = a_i + b_i + 1 \quad (i = 1, \cdots, r),$$

where a_i is the number of nodes in $[\lambda]$ to the right of (i, i) and (b_i) is the number of nodes in $[\lambda]$ below (i, i). Then

$$a_1 > a_2 > \cdots > a_r \ge 0, \quad b_1 > b_2 > \cdots > b_r \ge 0 \tag{3.6}$$

and

$$a_1 + \cdots + a_r + b_1 + \cdots + b_r + r = n.$$

The a_i, b_i $(i = 1, \cdots, r)$ are determined uniquely by (λ); and, conversely, (λ) is determined uniquely by r and the a_i, b_i $(i = 1, \cdots, r)$. This justifies the Frobenius symbol

$$\begin{pmatrix} a_1, \cdots, a_r \\ b_1, \cdots, b_r \end{pmatrix} \tag{3.7}$$

for (λ).

Now consider μ_h for $h \geq 0$. If $a_{i+1} < h \leq a_i$ $(i = 1, \cdots, r)$, then $\mu_h = i$ (we set $a_{r+1} = b_{r+1} = 0$). Similarly, if $h \leq 0$ then $\mu_h = j$, where $b_{j+1} < -h \leq b_j$.

These statements imply that

$$\mu_h - \mu_{h+1} = \begin{cases} 1 \text{ if } h = a_i & (i = 1, \cdots, r), \\ -1 \text{ if } h = -b_i - 1 & (i = 1, \cdots, r), \\ 0 \text{ otherwise.} \end{cases} \tag{3.8}$$

This establishes the "only if" part of the following theorem.

(3.9) *Let* $\Pi\, x_h^{\mu_h}$ *be a monomial of degree* n. *Then there exists a partition* (λ) *of* n *whose Young Diagram has this monomial for its content if and only if*

(i)
$$m_h = \mu_h - \mu_{h+1} = \begin{cases} 0, 1 & \textit{for } h \geq 0, \\ 0, -1 & \textit{for } h \leq 0, \end{cases}$$

and

(ii)
$$\Sigma m_h = 0 .$$

To establish the "if" part we denote by r the number of positive m_h, and define a_i and b_i by (3.8). Then the partition (λ) whose Frobenius symbol is $\begin{pmatrix} a_1, \cdots, a_r \\ b_1, \cdots, b_r \end{pmatrix}$ has content $\Pi\, x_h^{\mu_h}$. This is our first construction for obtaining $[\lambda]$ from its content. The second construction is given in the next section.

4. THE FUNDAMENTAL THEOREM. We will call a monomial *admissible* if it satisfies the conditions of (3.9). In this section we obtain a formula which determines the diagram $[\lambda]$ corresponding to an admissible monomial. The connection is made in terms of the first column hook lengths l_i.

The following theorem is almost obvious.

(4.1) *Let* (λ) *be a partition of* n *in the form* (3.1). *Then the last node in the* i-*th row of* $[\lambda]T^k$ *has content* x_{l_i} $(i = 1, \cdots, k)$, *and the last node in the first column of* $[\lambda]T^k$ *has content* x_1.

Let L be the first column or set of left end nodes of the rows of $[\lambda]$, and let R be the set of right end nodes of the rows of $[\lambda]$. Then clearly

$$C(LS) \cdot C[\lambda] = C[\lambda]S \cdot C(R) . \tag{4.2}$$

(This equation was suggested to the authors by J. S. Frame.)

Next, apply T^k to both sides of (4.2) and equate the contents of the results. First, we note that $C(LST^k) = x_0 x_1 \cdots x_{k-1}$ since the bottom node of $(LS)T^k$ lies in the zero diagonal (see (4.1)). Next,

$$C(RT^k) = \prod_{i=1}^{k} x_{l_i}$$

by (4.1); we denote this monomial by γ. Putting these results together and using (2.6) we have

$$C[\lambda]T^k(x_0x_1\cdots x_{k-1}) = C[\lambda]ST^k\gamma. \tag{4.3}$$

It is easy to verify that

$$T^k(x_0x_1\cdots x_{k-1}) = (T\,x_0)^k; \tag{4.4}$$

hence (4.3) can be written in the form

$$C[\lambda](T\,x_0)^k = C[\lambda]ST^k\gamma. \tag{4.5}$$

Finally, we may divide both sides of (4.5) by $C[\lambda]ST^k$, and we obtain our main result

$$\gamma = \frac{C[\lambda]}{C[\lambda]S}(T\,x_0)^k. \tag{4.6}$$

If in this formula $C[\lambda]$ is replaced by any admissible monomial $\theta = \Pi x_h^{\mu_h}$, we can calculate γ and therefore obtain (λ) in the following way. The integer k is determined by Formula (3.4) from θ. Then

$$\gamma = \frac{\theta}{\theta S}(T\,x_0)^k \tag{4.7}$$

contains only known terms and so can be calculated.

One further simplification is possible. We call the monomial

$$D[\lambda] = \frac{C[\lambda]}{C[\lambda]S}$$

the *trace* of $[\lambda]$, and it follows from (2.5) and (3.9) that

$$D[\lambda] = \prod_{h=-k}^{\lambda_1-1} x_h^{m_h}, \tag{4.8}$$

and thus we have

$$\gamma = \prod_{h=-k}^{\lambda_1-1} x_h^{m_h}(T\,x_0)^k. \tag{4.9}$$

In view of Formula (3.8), this provides a direct analytic connection between the Forbenius symbols a_i, b_i and the λ_i.

We close the section with an illustrative example. The monomial

$$\theta = x_{-4}x_{-3}^2x_{-2}^2x_{-1}^3x_0^3x_1^3x_2^2x_3^2x_4x_5x_6$$

is admissible. We have

$$\frac{\theta}{\theta S} = x_{-5}^{-1} x_{-4}^{-1} x_{-2}^{-1} x_1 x_3 x_6 .$$

According to (3.8), the Frobenius symbol for the corresponding partition (λ) is $\begin{pmatrix} 6, & 3, & 1 \\ 4, & 3, & 1 \end{pmatrix}$.

Applying our second method, we have $k = 5$ and hence

$$\begin{aligned} \gamma &= (x_{-5}^{-1} x_{-4}^{-1} x_{-2}^{-1} x_1 x_3 x_6)(T x_0)^5 \\ &= x_0^{-1} x_1^{-1} x_3^{-1} x_6 x_8 x_{11} x_0 x_1 x_2 x_3 x_4 \\ &= x_2 x_4 x_6 x_8 x_{11} . \end{aligned}$$

Thus

$$(l_1, l_2, l_3, l_4, l_5) = (11, 8, 6, 4, 2)$$

and

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (7, 5, 4, 3, 2).$$

5. THE MODULAR CASE. Let q be any positive integer, and let $\{y_0, y_1, \cdots, y_{q-1}\}$ be indeterminates. We define the q-*content* $C_q(i, j)$ of a node (i, j) to be y_h, where h is the smallest nonnegative integer congruent to $j - i$ modulo q; we define the q-content of finite subsets of $I \times I$ analogously. In this section we obtain several relations between q-content and q-cores.

We recall that a q-core is a Young diagram which has in it no hook of length q. The following criterion for q-cores was developed in [5] and [6].

(5.1) *A Young diagram* $[\lambda]$ *is a* q-*core if and only if every class of congruent* l_i*'s contains all smaller nonnegative integers congruent to the largest one in the class, the zero class being empty.*

The last proviso is obviously necessary since, if $l_i = sq$, then the rim of H_{i1} contains a q-hook.

(5.2) *The* q-*content of every* q-*hook is* $y_0 y_1 y_2 \cdots y_{q-1}$.

This follows immediately [4, Part II] from the periodicity of the operators T and S defined in (2.4).

It follows from (5.2) that two Young diagrams with the same q-core and the same weight n have the same q-content. Littlewood [3] proved the converse of this statement. In particular we have

(5.3) *Two* q-*cores* $[\lambda]$ *and* $[\lambda']$ *are equal if and only if their* q-*contents are equal.*

From (5.2) and (5.3) we see that

(5.4) *Two Young diagrams* $[\lambda]$ *and* $[\lambda']$ *have the same* q-*core if and only if the quotient of their* q-*contents is a power of* $y_0 y_1 \cdots y_{q-1}$.

We define the shift operators S and T on monomials in $y_0, \cdots, y_{q-1}$ just as for the nonmodular case (2.5), except that we replace y_q by y_0 and y_{-1} by y_{q-1}. Note that T^q and S^q are both equal to the identity operator on monomials in the y_h. Moreover, (5.5) $$(y_0 y_1 \cdots y_{q-1})T = (y_0 y_1 \cdots y_{q-1})S = y_0 y_1 \cdots y_{q-1}.$$

For any partition (λ) we have, from (4.8),

(5.6) $$D_q[\lambda] = \frac{C_q[\lambda]}{C_q[\lambda]S} = \Pi\, y_h^{n_h};$$

we call $D_q[\lambda]$ the q - *trace* of $[\lambda]$.

(5.7) *Two diagrams* $[\lambda]$ *and* $[\lambda']$ *have the same core if and only if they have the same* q - *trace. For any* $[\lambda]$ *the sum of the exponents in its* q - *trace is zero. Conversely, if* $\delta = \Pi y_h^{n_h}$ *and* $\Sigma n_h = 0$, *then there exists a unique core* $[\lambda]$ *having* δ *as its* q - *trace.*

The first statement follows from (5.4) and (5.5). The second statement follows from the fact that the monomials in the numerator and denominator of the second member of (5.6) have equal degree. The uniqueness part of the third statement follows from the first statement. We establish the existence as one step in giving a construction for $[\lambda]$ in terms of δ.

First, suppose $[\lambda]$ given and equate the q - contents of both sides of (4.2). Following the development of Section 4, this gives

(5.8) $$C_q[\lambda](Ty_0)^k = C_q[\lambda]ST^k\gamma_q,$$

where γ_q is the q - content of RT^k. We can now solve for γ_q:

(5.9) $$\gamma_q = D_q[\lambda](Ty_0)^k.$$

On the other hand we have, by the definition of R,

(5.10) $$\gamma_q = \prod_{h=1}^{q-1} y_h^{k_h}, \quad k = \Sigma k_h,$$

where k_h is the number of first column hooks of lengths congruent to h modulo q ($k_0 = 0$ by (5.1)). Now, by (5.1), we can reconstruct γ from γ_q if $[\lambda]$ is a core, that is, if the l_i congruent to h modulo q are

(5.11) $$h,\ h + q,\ \cdots,\ h + q(k_h - 1).$$

We replace $D_q[\lambda]$ in (5.9) by any δ of degree zero, and choose k as the smallest nonnegative integer for which $\delta(Ty_0)^k$ has all its exponents nonnegative. Then, by the minimality of k, $\delta(Ty_0)^{k-1}T$ has exactly one negative exponent, and this must be that of y_0. Hence

(5.12) $$\delta(Ty_0)^k = \prod_{i=1}^{q-1} y_i^{k_h}, \quad \Sigma k_h = k.$$

Now let $[\lambda]$ be the core for which $\gamma_q = \delta(Ty_0)^k$. We observe that for any power product θ in the y_h we have $\theta(Ty_0)(y_0^{-1}S) = \theta$; hence powers of Ty_0 can be cancelled from the right. In particular, by comparison of (5.9) and (5.12) we conclude that $\delta = D_q[\lambda]$. This completes the proof of (5.7).

Before turning to our construction for $[\lambda]$ from δ, we give further consideration to the determination of k. First, we observe that for any power product θ in the y_h we have

$$\theta(Ty_0)^q = \theta y_0 y_1 \cdots y_{q-1}. \tag{5.13}$$

Now, let $\delta = \Pi y_h^{n_h}$ have degree zero and let $-j = \min_h n_h$. Then $\delta(Ty_0)^{qj}$ has all exponents nonnegative, whereas $\delta(Ty_0)^{q(j-1)}$ has at least one negative exponent. Hence, $q(j-1) < k \leq qj$. The exact formula for k is

$$k = qj - s \tag{5.14}$$

where $-j = \min_h n_h$ and s is the smallest index for which $n_h = -j$. The value for s is obtained as the largest integer v for which $\delta(Ty_0)^{qj}(y_0^{-1}S)^v$ has all exponents nonnegative.

We illustrate with the example $q = 3$, $\delta = y_0^3 y_1^2 y_2^{-5}$. Here $j = 5$ and $s = 2$; so $k = 13$, and $\delta(Ty_0)^{13} = y_1^7 y_2^6$ determines the partition

$$(\lambda) = (7, 6^2, 5^2, 4^2, 3^2, 2^2, 1^2).$$

6. DIAGONAL HOOKS AND q-CONTENT. In this section we give a second construction for obtaining the q-core whose q-trace is a given power product δ of degree zero. This construction rests on the following property of the Frobenius symbol for a q-core.

(6.1) *Let* $\begin{pmatrix} a_1, \cdots, a_r \\ b_1, \cdots, b_r \end{pmatrix}$ *be the Frobenius symbol for a partition* (λ). *Then* $[\lambda]$ *is a q-core if and only if* (i) *each class of congruent* a_i*'s and each class of congruent* b_i*'s contains all smaller nonnegative integers congruent to the largest one in the class and* (ii) *for no* (i, j) *is* $a_i + b_j + 1$ *divisible by* q.

We prove this theorem by partitioning the hooks of $[\lambda]$ into three classes. Class 1 contains all hooks H_{ij} having $i \leq r$ and $j \leq r$; class 2, those having $i \leq r$ and $j > r$; and class 3, those having $j \leq r$ and $i > r$. Since r is the rank of $[\lambda]$, no hook has $i > r$ and $j > r$; hence every hook falls into at least one class.

We first consider an H_{ij} in class 1. We claim that

$$h_{ij} = a_i + b_j + 1 \quad \text{if } i \leq r \text{ and } j \leq r. \tag{6.2}$$

If $j = i$, this is true by definition of a_i and b_i. If $i < j$, then the distance from (i, j) to (i, i) is the same as that to (j, j), hence there are exactly a_i nodes of H_{ij} above and to the right of its diagonal node (j, j). Clearly, there are b_j nodes below the diagonal, and hence (6.2) holds in this case. A similar argument holds for $i > j$.

It follows at once from (6.2) that (6.1 ii) is a necessary and sufficient condition for the absence of hooks of length q (or of length divisible by q) in class 1.

The proof of Theorem 6.1 included the following lemma (see [1] and [6]).

(6.3) *For any* $[\lambda]$ *and for each* $i \leq k$, *the* l_i *numbers* $h_{i1}, \cdots, h_{i\lambda_i}, l_i - l_{i+1}, \cdots, l_i - l_k$ *are a permutation of* $1, \cdots, l_i$.

In other words, given i such that $l_i > q$, there exists a hook h_{ij} of length q if and only if no difference $l_i - l_k$ is equal to q. Moreover, if H_{ij} has its foot node in row h, then a study of its position in $[\lambda]$ shows that

(6.4) $$l_i - l_h < h_{ij} < l_i - (k - h + j - 2) \leq l_i - l_{h+1}.$$

If, in particular, H_{ij} is in class 2, then $h \leq r < j$ and it follows from (6.4) that

(6.5) $$l_i \geq h_{ij} + k.$$

On the other hand we have from (6.2) for $j = 1$ that

(6.6) $$l_i = a_i + k \quad (i = 1, \cdots, r).$$

From (6.5) and (6.6) we get

(6.7) $$a_i \geq h_{ij} \text{ whenever } i \leq r \text{ and } j > r.$$

Now suppose that $h_{ij} = q$ and H_{ij} is in class 2. Then $a_i - a_s = l_i - l_s$ for all s, and hence it follows from (6.3) that (6.1 i) does not hold.

Conversely, if (6.1 i) holds and $a_i \geq q$, there exists an s such that

$$a_i - a_s = l_i - l_s = q,$$

and hence there is no hook H_{ij} of length q. Thus we see that (6.1 i) is a necessary and sufficient condition for the absence of hooks of length q in class 2.

Interchanging the roles of rows and columns, we see that the part of (6.1 i) involving the b_i is a necessary and sufficient condition for the absence of hooks of length q in class 3. This completes the proof of (6.1).

(6.8) *Let* $[\lambda]$ *be a* q-*core with* q-*trace* $D_q[\lambda] = \Pi y_h^{n_h}$. *Then* n_h *is the sum of all* m_i *with* $i \equiv h \pmod q$, *and the nonzero summands of* n_h *all have the same sign. Moreover, the rank* r *of* $[\lambda]$ *is the sum of those* n_h *which are positive.*

The value given for n_h is an immediate consequence of the definitions of content and q-content (and is correct whether or not $[\lambda]$ is a q-core). Suppose next that two summands for n_h have unlike signs, say

$$m_s = 1, \; m_{-l} = -1, \text{ where } h \equiv s \equiv -l \pmod q.$$

Then, according to (3.8), there exist i and j such that $a_i = s$, $b_j = l - 1$, and hence $a_i + b_j + 1$ is divisible by q. But this is impossible for a q-core by (6.1 ii); we conclude that there is no cancellation in the sum of the n_h. Finally, the rank of r is by definition the number of positive m_h, and hence for q-cores it is the sum of positive n_h.

We now provide a second construction for the q-core $[\lambda]$ having a given q-trace $\delta = \Pi y_h^{n_h}$. If n_h is positive, then the r numbers a_i in the Frobenius symbol for $[\lambda]$ include $h, h + q, \cdots, h + (n_h - 1)q$; if n_h is negative, the r numbers b_j include $q - h - 1, 2q - h - 1, \cdots, -n_h q - h - 1$. Since r is the sum of the positive n_h and $-r$ is the sum of the negative n_h, this completely describes the Frobenius symbol for $[\lambda]$.

In particular, for the example $\delta = y_0^3 y_1^2 y_2^{-5}$ treated in the previous section, the a_i consist of 0, 3, 6, 1, 4 and the b_j of 0, 3, 6, 9, 12, and the Frobenius symbol is

$$\begin{pmatrix} 6, & 4, & 3, & 1, & 0 \\ 12, & 9, & 6, & 3, & 0 \end{pmatrix}.$$

We illustrate the results of this section and the previous one by giving a table for the case of 3 - cores, arranged according to the 3 - content γ_3 of the right-end nodes of the rows of $[\lambda]T^k$.

$[\lambda]$	γ_3	$D_3[\lambda]$	$C_3[\lambda]$	(l)	$\binom{a}{b}$
[1]	y_1	$y_0y_2^{-1}$	y_0	(1)	$\binom{0}{0}$
[2]	y_2	$y_1y_2^{-1}$	y_0y_1	(2)	$\binom{1}{0}$
[3, 1]	y_1^2	$y_1^{-1}y_2$	$y_0y_1y_2^2$	(4, 1)	$\binom{2}{1}$
$[1^2]$	y_1y_2	$y_0y_1^{-1}$	y_0y_2	(2, 1)	$\binom{0}{1}$
[4, 2]	y_2^2	$y_0^2y_1^{-1}y_2^{-1}$	$y_0^3y_1y_2^2$	(5, 2)	$\binom{3,\ 0}{1,\ 0}$
[5, 3, 1]	y_1^3	$y_0^{-1}y_1^2y_2^{-1}$	$y_0^3y_1^4y_2^2$	(7, 4, 1)	$\binom{4,\ 1}{2,\ 0}$
$[2, 1^2]$	$y_1^2y_2$	$y_0^{-1}y_1$	$y_0y_1^2y_2$	(4, 2, 1)	$\binom{1}{2}$
$[3, 1^2]$	$y_1y_2^2$	$y_0^{-1}y_2$	$y_0y_1^2y_2^2$	(5, 2, 1)	$\binom{2}{2}$
[6, 4, 2]	y_2^3	$y_0^{-1}y_1^{-1}y_2^2$	$y_0^3y_1^4y_2^5$	(8, 5, 2)	$\binom{5,\ 2}{2,\ 1}$

7. A MODIFIED PARTITION GENERATING FUNCTION. The significance of the partition generating function

$$P(x) = (1 - x)^{-1}(1 - x^2)^{-1}(1 - x^3)^{-1} \cdots \tag{7.1}$$

for the modular representation theory of the symmetric group was discussed in [2]. We now give a modification of (7.1) which is a generating function for the contents of all Young diagrams $[\lambda]$.

We write $(1 - y)^{-1}$ as an abbreviation for the formal power series $1 + y + y^2 + \cdots$. We next show that

$$(1 - Sx_0)^{-1}(1 - Sx_0x_1)^{-1}(1 - Sx_0x_1x_2)^{-1} \cdots = 1 + \sum_{n=1}^{\infty} \sum_{(\lambda)} C[\lambda], \tag{7.2}$$

where the inner summation is over all partitions (λ) of n.

We establish (7.2) by factoring $C[\lambda]$ in such a way as to show how it appears on the left-hand side. The content of the i - th row of $[\lambda]$ is clearly

$$x_{1-i} \cdots x_{\lambda_i - i} = (x_0 \cdots x_{\lambda_i - 1})S^{i-1}. \tag{7.3}$$

It follows from (7.3) that $C[\lambda]$ can be written in the form

$$C[\lambda] = (Sx_0 \cdots x_{\lambda_k - 1})(Sx_0 \cdots x_{\lambda_{k-1} - 1}) \cdots (Sx_0 \cdots x_{\lambda_1 - 1}). \tag{7.4}$$

Suppose that α_j of the λ_i are equal to j (j = 1, ⋯, n). Then

$$C[\lambda] = (Sx_0)^{\alpha_1} (Sx_0 x_1)^{\alpha_2} \cdots (Sx_0 x_1 \cdots x_{n-1})^{\alpha_n},$$

and in this form its identification with a unique term in the expanded form of the first member of (7.2) is obvious. Conversely, each term of the product is the content of some Young diagram. The extension of these ideas to the modular case is immediate.

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