ON THE NOTION OF BALANCE OF A SIGNED GRAPH

Frank Harary

This note deals with a generalization of linear graphs. The concepts which it introduces, as well as the manner in which it develops these concepts, were suggested by certain problems in social psychology. The sociometric structure of a group of persons is often represented by means of a square matrix M with a 0 diagonal and with 0's and 1's in the remaining positions. Let $M = \|m_{ij}\|$; then m_{ij} is 1 if person P_i likes person P_j, and is 0 otherwise. Such matrices correspond to irreflexive binary relations, or to directed graphs (digraphs). The motivation for defining signed graphs arose from the fact that psychologists have also employed square matrices with elements -1, 0, and 1 to represent disliking, indifference, and liking respectively. When a matrix of this sort is symmetric, it can be depicted by an ordinary linear graph which is modified by labelling some of its lines as positive, and the others as negative. Such a modified linear graph will be called a signed graph. If the structure matrix is not symmetric, it can still be represented by means of a directed graph rather than an ordinary graph. Some psychological interpretations of the theory of signed graphs will appear elsewhere.

The standard definitions used in the theory of linear graphs may be found in [3] or, with a social scientific bias, in [2]. However, for the sake of completeness, we shall include some definitions here. A (linear) *graph* G is a collection of n points $P_1, P_2, \cdots, P_n$ together with a given subset L of the set of all unordered pairs of distinct points. The pairs of points which occur in the set L will be called the *lines* of the graph. A *path* of G is a collection of lines of the form $A_1A_2, A_2A_3, \cdots, A_{m-1}A_m$, where the points $A_1, A_2, \cdots, A_m$ are distinct. The *cycle* $A_1A_2\cdots A_m$ of G consists of the path described above together with the line A_mA_1. Two points A and B of G are *adjacent* if the line AB is in G. A *complete* graph is one in which each point is adjacent to every other point. A graph is *connected* if every pair of distinct points is joined by a path. A *tree* is a connected graph without cycles.

A *signed graph* G, or briefly an s-*graph*, consists of a set E of n points $P_1, P_2, \cdots, P_n$ together with two disjoint subsets L^+, L^- of the set of all unordered pairs of distinct points. The elements of the sets L^+, L^- are called *positive lines* and *negative lines* respectively. A *positive cycle* of an s-graph is one in which the number of negative lines is even; a *negative cycle* is not positive. Similarly, the sign of a path is the product of the signs of its lines.

An s-graph is in *balance* if all its cycles are positive. We shall develop a characterization of balance for an abritrary s-graph, and in addition we will describe a procedure for enumerating s-graphs. We begin with a structure theorem for complete balanced s-graphs.

THEOREM 1. *A complete* s-*graph* G *is balanced if and only if its point set* E *is partitioned into two disjoint subsets* E_1 *and* E_2, *one of which may be empty, such that all lines between points of the same subset are positive and all lines between points of the two different subsets are negative.*

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Proof of necessity. Let A_1 be any point of G, E_1 the set containing A_1 and all points positively adjacent to A_1, and E_2 the collection of all other points, i.e., all points negatively adjacent to A_1. Then the sets E_1 and E_2 are disjoint, and $E = E_1 \cup E_2$.

Any two distinct points B_1, C_1 of E_1 are positively adjacent. For, if one of these two points is A_1, this is the construction of the set E_1. But if neither of the points is A_1, then the cycle of length 3 containing A_1, B_1, and C_1 contains the positive lines A_1B_1 and A_1C_1. Since this cycle is positive, the line B_1C_1 is also positive.

Any two distinct points B_2, C_2 of E_2 are also positively adjacent. This is seen at once from the fact that the positive cycle $A_1B_2C_2$ contains the negative lines A_1B_2 and A_1C_2. Finally, any line B_1B_2, where $B_1 \in E_1$ and $B_2 \in E_2$, is similarly observed to be negative.

Proof of sufficiency. Suppose that the conditions of the theorem are satisfied. Since every cycle in G contains an even number of $E_1 - E_2$ lines, every cycle is positive.

A psychological interpretation of Theorem 1 is that a "balanced group" consists of two highly cohesive cliques which dislike each other.

It follows at once from Theorem 1 that the number of different balanced complete s-graphs of n points is equal to the number of partitions of n into two non-negative summands, that is, to $1 + [n/2]$.

A *subgraph* of a graph G is a graph whose points and lines lie in G. It is convenient to call the following remark on subgraphs a lemma.

LEMMA. *Every subgraph of a balanced* s-*graph is balanced.*

Proof. Each cycle of the subgraph is a cycle of the balanced graph and is therefore positive.

THEOREM 2. *An* s-*graph is balanced if and only if for each pair of distinct points* A, B *all paths joining* A *and* B *have the same sign.*

Proof of necessity. Consider any two paths α_1, α_2 joining A and B. The deletion of their common lines (if any) leads to a collection of line-disjoint cycles. Each of these cycles z consists of a subpath of α_1 and a subpath of α_2. Since z is positive, these two subpaths have the same sign. Collecting all such subpaths together with the common lines, one sees that α_1 and α_2 have the same sign.

Proof of sufficiency. Since all paths joining A and B have the same sign, every cycle containing A and B must be positive. But A and B are arbitrary points. Hence all cycles are positive and the s-graph is balanced.

We can now prove an extension of Theorem 1 to s-graphs which are not necessarily complete.

THEOREM 3. *An* s-*graph* G *is balanced if and only if its point set* E *can be partitioned into two disjoint subsets* E_1, E_2 *in such a way that each positive line of* G *joins two points of the same subset and each negative line joins two points of different subsets.*

Proof of sufficiency. One may extend the given s-graph G to a complete balanced s-graph as follows. Take each pair of non-adjacent points A, B of G. If they are in the same subset, draw AB as a positive line; if in different subsets, draw AB as a negative line. By Theorem 1, the resulting complete s-graph is balanced. Hence by the lemma, G is also balanced.

Proof of necessity. The proof is by induction on the number of lines of the balanced s-graph G. This is accomplished by showing that one additional line of appropriate sign can be added to join a pair of non-adjacent points of G in such a way that the resulting s-graph is still balanced.

There is no loss of generality if we regard G as connected. Let A, B be any pair of non-adjacent points. By Theorem 2, all paths joining A and B have the same sign. Draw the line AB of the same sign as each of these paths. Then all the new cycles thus introduced are positive, and the resulting s-graph is balanced. Once G is completed, the necessity of the condition follows from Theorem 1.

Combining Theorems 1 and 3, one sees that a necessary and sufficient condition for an s-graph to be balanced is that it be a subgraph of a complete balanced s-graph. Note that any s-tree, having no cycles, is in balance.

Two s-graphs G and G' are *isomorphic* if there exists a one-to-one correspondence between their point sets E and E' which preserves adjacency and signs. Two s-graphs are *different* if they are not isomorphic. The number of different s-graphs having p points and k lines can readily be determined by using the formula from [1] for the counting polynomial $g_p(x)$ for all graphs of p points. The coefficient of x^k in this polynomial is the number of ordinary graphs with p points and k lines. The figure counting series for ordinary graphs is $1 + x$; the $1 = x^0$ denoting lack of adjacency of a pair of points and hence 0 lines, and the $x = x^1$ standing for adjacency. If one replaces $1 + x$ in the formula for $g_p(x)$ by $1 + x + y$, then the counting polynomial for all s-graphs of p points is obtained. The 1, x, and y represent non-adjacency, positive adjacency, and negative adjacency, respectively. In the resulting polynomial the coefficient of $x^{k_1}y^{k_2}$ is the number of different s-graphs with p points, k_1 positive lines, and k_2 negative lines. To illustrate, the counting polynomial for the s-graphs of three points is

$$1 + x + y + x^2 + xy + y^2 + x^3 + x^2y + xy^2 + y^3,$$

while that of four points is

$$\begin{gathered} 1 + x + y \\ + 2x^2 + 2xy + 2y^2 \\ + 3x^3 + 4x^2y + 4xy^2 + 3y^3 \\ + 2x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 2y^4 \\ + x^5 + 2x^4y + 4x^3y^2 + 4x^2y^3 + 2xy^4 + y^5 \\ + x^6 + x^5y + 2x^4y^2 + 2x^3y^3 + 2x^2y^4 + xy^5 + y^6. \end{gathered}$$

These results are readily generalized to the variation of s-graph in which more than one line is permitted to join the same pair of points. With the previous definition of a positive cycle understood to apply also to cycles of length 2, the statement of Theorem 3 still holds. The number of s-trees may be readily expressed in terms of the number of oriented trees, and will be given elsewhere.

REFERENCES

1. F. Harary, *The number of linear, directed, rooted, and connected graphs*, to appear in Trans. Amer. Math. Soc., 1955.

2. F. Harary and R. Z. Norman, *Graph theory as mathematical model in social science*, Research Center for Group Dynamics, Monograph No. 2, Ann Arbor, 1953.

3. D. König, *Theorie der endlichen und unendlichen Graphen*, Leipzig, 1936 (reprinted New York, 1950).

University of Michigan